Projection Singularities of Extremals for Planar Systems

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Abstract

For a generic single input planar control system we are concerning with the set of extremals for the minimum time problem. We study some singularities of the projection of the extremals's locus from the cotangent bundle to $\mathbf{R}^2$. The structure of this locus near the origin is showed in details and sufficient conditions under which it is a piecewise $\mathcal{C}^1$ manifold with boundary are given. Finally we give examples of cusps singularities that happen in presence of overlap curves of the optimal synthesis.

1 Introduction

Consider the control system:

$$\left\{\begin{array}{l} \dot{x} = F(x) + uG(x), \\ x \in \mathbf{R}^2, \quad F, G \in \mathcal{C}^3(\mathbf{R}^2, \mathbf{R}^2), \qquad \textit{[1.1]} \\ F(0) = 0, \quad |u| \leq 1 \end{array}\right. \tag{1}$$

Let $\mathcal{R}(\tau) = \{x \in \mathbf{R}^2 : \exists$ a trajectory of (1) s.t. $\gamma(0) = 0, \ \ \gamma(t) = x$ for some $t \leq \tau\}$ be the reachable set within time τ from the origin. We refer to [7] for basic theory. In the framework of [2, 5, 6], we are faced with the problem of reaching (under generic conditions on F and G) every point of $\mathcal{R}(\tau)$ in minimum time. The structure of optimal trajectories has been studied also in [8, 9].

For systems of the kind (1) and for the minimum time problem from the origin, the Pontryagin Maximum Principle states that if (γ, u) is an optimal trajectory–control pair on some interval $[0, t^*]$, then there exists a non trivial absolutely continuous function $p : [0, t^*] \to \mathbf{R}_2$ (row vectors) such that for a.e. $t \in [0, t^*]$ we have:

i) $\dot{p}(t) = -p(t) \cdot (\nabla F + u(t)\nabla G)(\gamma(t))$;

ii) $p(t) \cdot (F + u(t)G)(\gamma(t)) = \max_{|w| \leq 1} p(t) \cdot (F + wG)(\gamma(t))$;

iii) $p(t) \cdot (F + u(t)G)(\gamma(t)) = const \geq 0$.

We call extremals the trajectories of (1) for which there exists a p satisfying **i) ii) iii)**. In this case we say that (γ, p) is an extremal pair.

Consider the following lift of $\mathcal{R}(\tau)$:

$N_0 = \{(x, \lambda) \in \mathbf{R}^2 \times S^1$: there exists a time $t \leq \tau$ and an extremal pair $(\gamma, p) : \ [0, t] \to \mathbf{R}^2 \times \mathbf{R}_2$ such that $\gamma(0) = 0$, $\gamma(t) = x$, $p(0) = \lambda\}$.

In this paper we are going to study the locus N_0 to show that some singularities of the optimal synthesis of this problem (see [2, 5, 6]) correspond to singularities of the projection $N_0 \longrightarrow \mathcal{R}(\tau)$ given by the map $(x, \lambda) \mapsto x$. Moreover we show that the lift of the extremal trajectories after the first switching point are the iso–surfaces of N_0 and that (x, λ), (x', λ) belong to N_0 ($x \neq x'$) only if x and x' are reached by the same extremal trajectory (at different time) or they belong to a "flat" region determined by singular trajectories (see Theorem 1 and Lemma 1). Using this arguments we can argue in which cases N_0 is a $\mathcal{C}^1$ manifold (with boundary). More details are given in a neighborhood of the origin.

The singularities of the set of extremals for general optimization problems has been studied also in [4]. In [1] a similar investigation is developed.

In section 2, we review the basic definitions and some related results from [2, 3, 5, 6]. Moreover we give a generalization of these results from the optimal to the extremal case.
In section 3, we prove a Theorem that gives us important information about the structure of N_0. Moreover we prove that in a neighborhood of the origin N_0 is a piecewise $\mathcal{C}^1$ manifold with boundary and we give some examples of regions whose projection on $\mathcal{R}(\tau)$ is not trivial.

In section 4, we study when extremal pairs intersect each others in the cotangent bundle.
In section 5, we discuss the lift of N_0 in the space (x, λ, t) (where t is the extremal time) that permits to understand under which conditions N_0 is a $\mathcal{C}^1$ manifold with boundary.

2 Preliminaries

In [5] it was shown that, under generic conditions, the problem of reaching in minimum time every point of the reachable set for the system (1) admits a regular synthesis. Moreover it was shown that $\mathcal{R}(\tau)$ can be s-tratified in a finite number of embedded submanifolds of dimension 2,1 and 0 such that the optimal trajectories are generated by a discontinuous feedback $u(x)$ such that:

- on the regions of dimension 2, we have $u = \pm 1$,
- on the regions of dimension 1, called frame curves (FC in the following), $u(x) = \pm 1$ or $u(x) = \varphi(x)$ (where $\varphi(x)$ is a feedback control that depends on F, G and on their Lie bracket $[F, G]$, see [5]).

The submanifold of dimension 0 are called frame points (FP in the following). In [6] it was provided a complete classification of all types of FP and FC for the optimal case.

To compute the locus N_0 we have to consider all the extremal trajectories. The algorithm $\mathcal{A}$ of [5] can be modified to produce all the extremals simply by not performing "cuts". A program of classification of all the FC and FP can be outlined also in the this case (see [1] for details). The basic differences from the optimal case are the following:

- there are not any K FC and any $(K, .)$ FP.
- there are new FC and FP that are not considered in the optimal case because they do not satisfy higher order conditions.

From mow on we will make the generic conditions of [2, 5, 6].

3 The locus N_0 in a neighborhood of the origin

Consider the projector $\Pi : (x, \lambda) \in \mathbf{R}^2 \times S^1 \mapsto x \in \mathbf{R}^2$. Clearly we have $\mathcal{R}(\tau) = \Pi(N_0)$. If we set $\Lambda^0(x) = \{\lambda \in S^1 : (x, \lambda) \in N_0\}$ we can regard N_0 as a trivial fiber bundle with base $\mathcal{R}(\tau)$ and fiber $\Lambda^0(x)$ i.e. $N_0 = \bigcup_{x \in \mathcal{R}(\tau)} (x, \Lambda^0(x))$. Obviously we have $(x, \Lambda^0(x)) = \Pi^{-1}(x) \cap N_0$.

Let $\gamma^+ : [0, t^+] \to \mathbf{R}^2$ be the curve starting from the origin with control $+1$, where t^+ is the last time in which it is extremal (in the case in which the trajectory fails to be extremal before τ) or it is equal to τ (in the opposite case). In [2] it was defined the function $\theta^+ : [0, t^+] \to [-\pi, \pi]$ associated to the curve γ^+ by $\theta^+(t) = \arg(\bar{v}(0), \bar{v}(t))$, where $\bar{v}(t) := v(G(\gamma(t)), t, 0)$ is the solution in the case $v_0 = G(\gamma(t))$ and $t_0 = 0$ to the Cauchy problem:

$$\begin{cases} \dot{v}(v_0, t_0; t) & = & (\nabla F + \nabla G)(\gamma^+(t)) \cdot v(v_0, t_0; t) \\ v(v_0, t_0; t_0) & = & v_0. \end{cases}$$

Moreover the following times were single out:

$$\begin{aligned}
&t_0 = t'_0 = 0 \\
&t_1 = \inf\{t > 0 : \ \theta^+ \text{ has a local max. at } t, \\
&\qquad\qquad\qquad \theta^+(t) > 0\} \\
&t_i = \inf\{t > t_{i-1} : \ \theta^+ \text{ has a local max. at } t, \\
&\qquad\qquad\qquad \theta^+(t) > \theta^+(t_{i-1})\} \\
&t'_1 = \inf\{t > 0 : \ \theta^+ \text{ has a local min. at } t, \\
&\qquad\qquad\qquad \theta^+(t) < 0\} \\
&t'_i = \inf\{t > t'_{i-1} : \ \theta^+ \text{ has a local min. at } t, \\
&\qquad\qquad\qquad \theta^+(t) < \theta^+(t_{i-1})\} \\
&s_i = \max\{t \in [t_{i-1}, t_i] : \ \theta^+(t) = \theta^+(t_{i-1})\} \\
&s'_i = \max\{t \in [t'_{i-1}, t'_i] : \ \theta^+(t) = \theta^+(t'_{i-1})\}.
\end{aligned}$$

Similarly it was defined the function θ^- associated to γ^-. The structure of N_0 is contained in the following Theorem:

Theorem 1 *Chose a coordinate system such that* $G(0) = (0, 1)$.

A) *Let* $x \in \mathcal{R}(\tau) \setminus (supp(\gamma^+) \cup supp(\gamma^-))$ *and fix an extremal trajectory that reaches* x *at some time* $\bar{t}$*:*

$$\gamma : [0, \bar{t}] \to \mathcal{R}(\tau)$$
$$\gamma(0) = 0, \quad \gamma(\bar{t}) = x.$$

Then there exists a unique covector $p : [0, \bar{t}] \to \mathbf{R}^2$ *s.t.* $|p(0)| = 1$ *and* (γ, p) *is an extremal pair in* $[0, \bar{t}]$*. Moreover, let* $t_0(\gamma) \in]0, t^+[$ *be the first switching time of* γ*, say from constant control +1 to constant control -1 or* $\varphi(\gamma(t))$*, then we have:*

$$\Lambda^0(x) = \begin{cases} \theta^+(t_0(\gamma)) & \text{if } t_0(\gamma) \in [s_i, t_i] \\ \pi + \theta^+(t_0(\gamma)) & \text{if } t_0(\gamma) \in [s'_i, t'_i] \end{cases} \qquad [\,qwd] \quad (2)$$

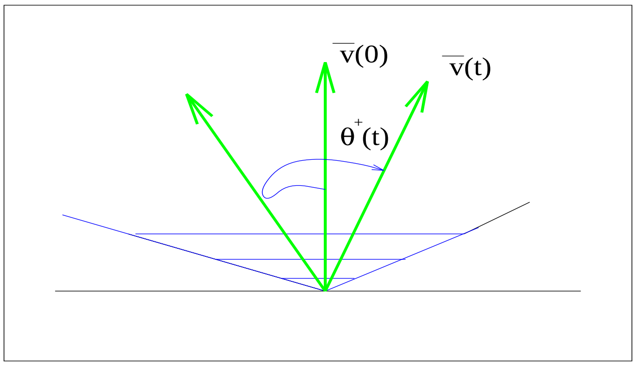

Figure 1: Proof of Theorem 1
[vvg]

B) $\Lambda^0(0) = [-\pi, \pi]$

C) *Let* $x \in$ *supp*$(\gamma^+) \setminus \{0\}$ *and let* $t(x) \in]0, t^+]$ *be the first time such that* $x = \gamma^+(t(x))$*, then:*

$$\Lambda^0(x) = \begin{cases} [\theta^+(t(x)), \pi + \theta^+(t'_{i-1})] & \text{if } t(x) \in]s_i, t_i] \\ [\theta^+(t_i), \pi + \theta^+(t'_{i-1})] & \text{if } t(x) \in]t_i, s'_i] \\ [\theta^+(t_i), \pi + \theta^+(t(x))] & \text{if } t(x) \in]s'_i, t'_i] \\ [\theta^+(t_i), \pi + \theta^+(t'_i)] & \text{if } t(x) \in]t'_i, s_{i+1}] \end{cases}$$

[N0shape]

Proof of Theorem 1 Let us first prove **A)**. By definition p exists. We have to prove the uniqueness. Since γ is extremal, we know that there exists a non trivial covector $p : [0, \bar{t}] \to \mathbf{R}^2$ s.t. for a.e. $t \in [0, t_0]$:

$$\dot{p}(t) = -p(t) \cdot \nabla Y(\gamma(t))$$
$$p(t) \cdot G(\gamma(t)) \geq 0.$$

Using the definition of v and by the property $p(t) \cdot v(v_0, t_0; t)$ =const (see [2]) PMP implies that for a.e. $t \in [0, t_0]$:

$$0 \leq p(t) \cdot G(\gamma(t)) = \quad [\ ab] \qquad (3)$$
$$= p(t) \cdot v(G(\gamma(t)), t; t) =$$
$$= p(0) \cdot v(G(\gamma(t)), t; 0) =$$
$$= p(0) \cdot \bar{v}(t).$$

Now $\bar{v}(t)$ is a vector that satisfy $\arg(\bar{v}(0), \bar{v}(t)) = \theta^+(t)$, where θ^+ is a function whose range reaches π exactly at time t^+ (see [2]). So with our local coordinates we have the situation of fig. 1. All the $p(0)$ in the sketched region satisfy the condition 3. Now

- if $t_0 \in [s_i, t_i[$ (resp. $t_0 \in [s'_i, t'_i[$):

  1. from proposition 3.1 of [2] at time t_0 we have a YX switching;
  2. $\dot{\theta}(t_0) > 0$ (resp. < 0); *[Aa]* (4)
  3. formula (3) is satisfied with the equality at time t_0 i.e. $p(0) \cdot \bar{v}(t_0) = 0$ *[Bb]* (5)

- if $t_0 = t_i$ (resp. t'_i) then

  1. from proposition 3.1 of [2] at time t_0 we have a YX switching;
  2. θ has a maximum (resp. a minimum) at this time; *[Cc]* (6)
  3. $p(0) \cdot \bar{v}(t_0) = 0.$ *[Dd]* (7)

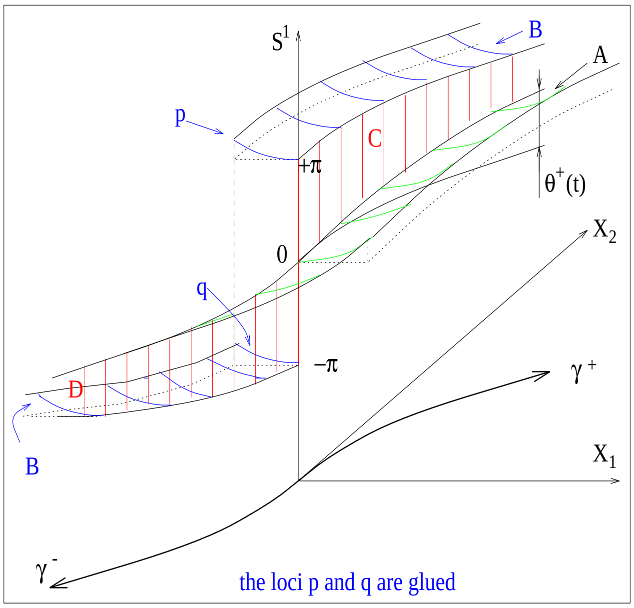

Figure 2: N_0 in a neighborhood of the origin
[origink]

Note that Proposition (3.1) of [2] excludes that $t_0 \in]t_i, s'_i[$ or $t_0 \in]t'_i, s_{i+1}[$. Formulas (4, 5) in the first case and (6, 7) in the second select exactly the $p(0)$ given by the right hand side of formula (2) that includes both cases. Now, since the evolution equation of p **i)** satisfies the Caratheodory conditions, $p(t)$ is unique, and it follows (2).

Because of the request "a.e." in the PMP, in the origin every $p(0)$ is allowed. By normalizing we have $\Lambda^0(0) = [-\pi, \pi]$. this proves **B)**.

By the same argument of the proof of formula (2) we have that for a.e. $t \in]0, t(x)]$ PMP says that:

$$0 \leq p(0) \cdot \bar{v}(t) \quad [\ poi] \qquad (8)$$

Again all the $p(0)$ in the sketched region of fig. 1 satisfy this condition. From the definition of the times

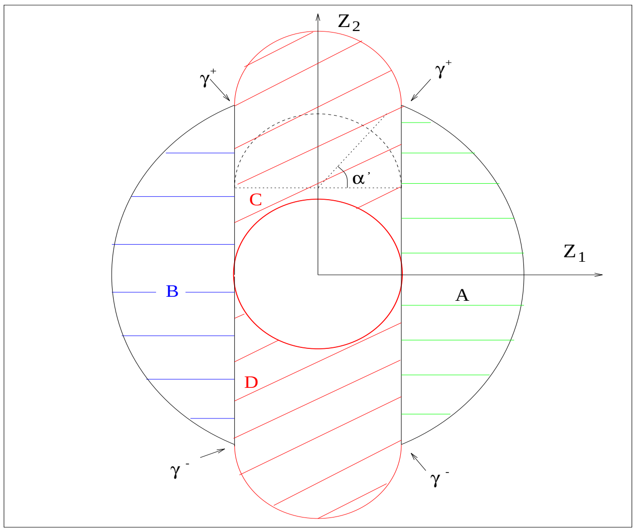

Figure 3: The set R

[Rk]

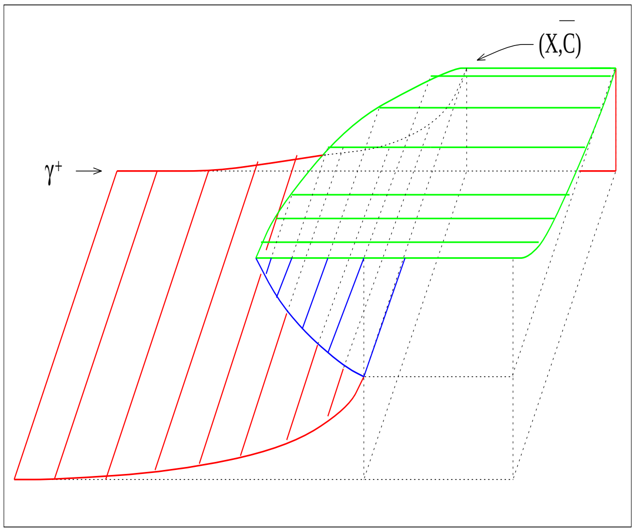

Figure 4: N_0 in a neighborhood of a cusp point

[cuspk]

t_i, s_i, t'_i, s'_i it follows the expression for $\Lambda^0(x)$ in the case $x \in \operatorname{supp}(\gamma^+) \setminus \{0\}$, that concludes the proof. ■

<u>*Observation*</u> **1** A similar Theorem can be written for the trajectory γ^- that starts from the origin with control -1.

Let $I(0)$ be a small neighborhood of the origin. The shape of $N_0|_{I(0)}$ can be obtained from Theorem 1 and it is drawn in fig. 2.

Now we will describe a piecewise $\mathcal{C}^1$ diffeomorphism from $N_0|_{I(0)}$ to $R := U \setminus B(0,1) \mapsto \mathbf{R}^2$ where U is a connected open region of $\mathbf{R}^2$ containing an open ball with center in 0 and radius 1 (see fig. 3). This proves that $N_0|_{I(0)}$ is a piecewise $\mathcal{C}^1$ manifold with boundary. In the following the subscript r will indicate the restriction to $I(0)$.

Chose a local system of coordinates such that $G(0) = (0,1)$ and let $x_1^+(x_2)$ (resp. $x_1^-(x_2)$) the support of γ_r^+ (resp. γ_r^-) expressed as a function of x_2. Moreover let (z_1, z_2) be the Cartesian coordinates of the plane containing R. Call O the region of N_0 identified by $\Pi(O) = 0$. Any point $y \in O$ is uniquely identified by an angle $\alpha \in [-\pi, \pi]$. Consider the map $\xi_O : O \to R$ defined by:

$$\alpha \mapsto (z_1(\alpha), z_2(\alpha)) = (\cos\alpha, \sin\alpha). \tag{9}$$

Let U_A and U_B the two connected components of $I(0) \setminus (\operatorname{supp}(\gamma_r^+) \cup \operatorname{supp}(\gamma_r^-))$, chosen is such a way U_A lies on the right of γ^+. Call A the region of N_0 identified by $\Pi(A) = U_A$. Any point $y \in A$ is uniquely identified by its projection $x = \Pi(y)$. Let $(x_1,\ x_2)$ the Cartesian coordinates of $x \in U_A$. Consider the map $\xi_A : A \to R$ defined by:

$$\begin{aligned}(x_1, x_2) \ \mapsto\ & (z_1(x_1,x_2), z_2(x_1,x_2)) = \\ & = \begin{cases} (x_1 + 1 - x_1^-(x_2), x_2) & \text{if } x_2 \le 0 \\ (x_1 + 1 - x_1^+(x_2), x_2) & \text{if } x_2 \ge 0 \end{cases}.\end{aligned}$$

Call C the region of N_0 identified by $\Pi(C) = \operatorname{supp}\ (\gamma_r^+) \setminus \{0\}$. Any $y \in C$ is uniquely identified by a couple (x, α) where $x \in \operatorname{supp}(\gamma_r^+) \setminus \{0\}$ and $\alpha \in [\theta^+(t), \pi]$. Let $t(x)$ the time s.t. $x = \gamma_r^+(t(x))$. Consider the map $\xi_C : C \to R$ defined by:

$$(x, \alpha) \mapsto (z_1(x,\alpha), z_2(x,\alpha)) = (\cos\alpha', x_2 + \sin\alpha')$$
$$\text{where } \alpha' = \frac{\pi}{\pi - \theta^+(t(x))}(\alpha - \theta^+(t(\ x)).$$

In similar way we can define the maps ξ_B, ξ_D. The map:

$$\xi(y) = \begin{cases} \xi_O(y) & \text{if } y \in O \\ \xi_A(y) & \text{if } y \in A \\ \xi_B(y) & \text{if } y \in B \\ \xi_C(y) & \text{if } y \in C \\ \xi_D(y) & \text{if } y \in D \end{cases} \tag{10}$$

is a piecewise $\mathcal{C}^1$–diffeomorphism of $N_0|_{I(0)} \to R$.

<u>*Observation*</u> **2** In the interior of every region this d-iffeomorphism is $\mathcal{C}^\infty$. N_0 looses to be $\mathcal{C}^1$ on the $A-C$, $C-B$, $A-D$ and $D-B$ edges that on R correspond all to $\gamma_r^+ \cup \gamma_r^-$.

Another region of N_0 that does not project trivially on $\mathbf{R}^2$ is a neighborhood of a singular point of the kind

$(X,\bar{C})$ (see [1]) where it generates a cusp. (fig. 4). This singularity corresponds to a local point (X,K) in the optimal synthesis (where an overlap curve starts). This is a general fact: *a cusp point of N_0 is associated to every point of the optimal synthesis in which an overlap curve starts* (see [1]). Notice that the fold points of the cusp are removed in the optimal synthesis.

4 Intersection of extremal pairs in the cotangent bundle

Define the loci:

$\mathcal{K}_1 =\{(x,\lambda) \in N_0$: there exist two times $t_1 < t_2 \leq \tau$ and an extremal pair (γ,p) such that $\gamma(t_1) = \gamma(t_2) = x$, $p(0) = \lambda\}$,

$\mathcal{K}_2 =\{(x,\lambda) \in N_0$: $\exists\ \varepsilon > 0$, three times $t_0 < t_1 \leq t_2 \leq \tau$ and two extremal pairs (γ_1,p_1), (γ_2,p_2) such that $\gamma_1(t_1) = \gamma_2(t_2) = x$, $\gamma_1|_0^{t_0} = \gamma_2|_0^{t_0}$, $p_1(0) = p_2(0) = \lambda$, the trajectory coinciding with γ_1 (and γ_2) in the interval $[0,t_0]$ and corresponding to the singular control φ in $[t_0,t_0+\varepsilon[$ is extremal$\}$.

Note that $\mathcal{K}_1$ and $\mathcal{K}_2$ are not necessarily disjoint. Moreover we can have $\mathcal{K}_1=\emptyset$ or $\mathcal{K}_2=\emptyset$. In particular both of them are empty if τ is sufficiently small. We have the following:

Lemma 1 *Let x_1, x_2 be two points of $\mathcal{R}(\tau)$, then (x_1,λ) and (x_2,λ) belong to N_0 iff one of the following conditions holds:*

C1 *there exists an extremal pair (γ,p) s.t. $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$, $p(0) = \lambda$ for some t_1 ,$t_2 \leq \tau$;*

C2 *there exist $\varepsilon > 0$, three times t_0, t_1, $t_2 \leq \tau$, ($t_0 < t_1$, $t_0 < t_2$) and two extremal pairs (γ_1,p_1), (γ_2,p_2) such that $\gamma_1(t_1) = x_1$, $\gamma_2(t_2) = x_2$, $p_1(0) = p_2(0) = \lambda$, $\gamma_1|_0^{t_0} = \gamma_2|_0^{t_0}$, the trajectory coinciding with γ_1 (and γ_2) in the interval $[0,t_0]$ and corresponding to the singular control φ in $[t_0,\varepsilon[$ is extremal.*

[usa-l-iff]

Proof of lemma 1 The proof that the conditions are sufficient is obvious. Now we will prove that they are necessary. Let $(\gamma_1,p_1):[0,t_1]\to\mathbf{R}^2\times\mathbf{R}_2$ and $(\gamma_2,p_2):[0,t_2]\to\mathbf{R}^2\times\mathbf{R}_2$ be the extremal trajectories such that $\gamma_1(t_1)=x_1$, $\gamma_2(t_2)=x_2$, $p_1(0)=p_2(0)=\lambda$. By Theorem 1 and from the strict monotonicity of θ^+ in the intervals $[s_i,t_i[$ and $[s_i',t_i'[$ (that follows from the generic assumption of [2]) we can conclude that:

1) if γ_1 and γ_2 switch at least one time then (x_1,λ), $(x_2,\lambda)\in N_0$ implies that γ_1 and γ_2 switch the first time at the same point. **C1** corresponds to the case in which they are reached by the same extremal trajectory. **C2** is the only possible case in which there is not forward uniqueness for the optimal feedback (see [2]);

2) if γ_1 has no switching and γ_2 has at least one switching then $\gamma_2|_{[0,t_1]}=\gamma_1$ and **C1** holds;

3) if γ_1 and γ_2 have no switching then either γ_1 is the restriction of γ_2 or viceversa, so we are in case **C1**.

This concludes the proof. ■

Theorem 2 *Let (x,λ) belong to N_0, then the extremal pair (γ,p) such that $\gamma(t)=x$, $p(t)=\lambda$ for some $t\leq\tau$ is unique iff $(x,\lambda)\notin\mathcal{K}_1\cup\mathcal{K}_2$. [usa-t-2]*

Proof of Theorem 2 By definition of the loci $\mathcal{K}_1$ and $\mathcal{K}_2$, if (γ,p) is unique then $x\notin\mathcal{K}_1\cup\mathcal{K}_2$. Now we will prove the other implication. Assume that there exist two distinct extremal pairs (γ_1,p_1), (γ_2,p_2) such that $\gamma_1(t_1)=\gamma_2(t_2)=x$ for some t_1, t_2 and $p_1(0)=p_2(0)=\lambda$. With arguments similar to the proof of Lemma 1 we can conclude that $x\in\mathcal{K}_1\cup\mathcal{K}_2$ (in this case we have $x_1=x_2$ but two different extremal pairs; more precisely if **C1** (resp. **C2**) holds then $x\in\mathcal{K}_1$ (resp. $x\in\mathcal{K}_2$)). ■

5 The lift of N_0 in the (x,λ,t) space.

Consider the following lift of N_0 in the (x,λ,t) space:

$\bar{N}_0=\{(x,\lambda,t)\in\mathbf{R}^2\times S^1\times[0,\tau]$: there exists an extremal pair (γ,p) : $[0,t]\to\mathbf{R}^2\times\mathbf{R}_2$ such that $\gamma(0)=0$, $\gamma(t)=x$, $p(0)=\lambda\}$.

We have the following:

Lemma 2 *Let (x,λ,t) belong to $\bar{N}_0$, then, generically, the extremal pair (γ,p) such that $\gamma(t)=x$, $p(t)=\lambda$ is unique. [usa-l-p2]*

Proof of lemma 2 By Theorem 2 we may consider only the regions $\mathcal{K}_1$ and $\mathcal{K}_2$. If $x\in\mathcal{K}_1$ then **C1** is satisfied but with $t_1 \neq t_2$. If $x \in \mathcal{K}_2$ we may assume that t_1 and t_2 are different by genericity. ■

By this Lemma it is clear that, generically, in the (x, λ, t) space no extremal trajectories intersect each other. In fact a more detailed local analysis (similar to the one of section 3 (see [1])) shows that holds:

Theorem 3 *Generically $\bar{N}_0$ is a piecewise $\mathcal{C}^1$ manifold with boundary.*

Now Theorem 2 and Lemma 2 show that in fact the projection $\bar{N}_0 \to N_0$ is singular only on the set $\mathcal{K}_1 \cup \mathcal{K}_2$. But $\mathcal{K}_2$ is a subset of a "flat region" because of the following:

Lemma 3 *Let $(x, \lambda) \in \mathcal{K}_2$ and assume that there exists $\varepsilon > 0$ such that $\gamma_1|_{[t_0,t_0+\varepsilon]}$, $\gamma_2|_{[t_0,t_0+\varepsilon]}$, correspond to the singular control φ then there exists $\delta > 0$ such that for every $(x', \lambda') \in N_0$ satisfying $|\lambda' - \lambda| < \delta$ and $|x' - x| < \delta$ we have $\lambda' = \lambda$. [usa-l-3]*

Proof of lemma 3 Since $(x, \lambda) \in \mathcal{K}_2$ we know that there exists a time t_0 such that $\gamma_1|_0^{t_0} = \gamma_2|_0^{t_0}$ and the trajectory coinciding with γ_1 (and γ_2) in the interval $[0, t_0]$ and corresponding to the singular control φ in $[t_0, t_0 + \varepsilon[$ is extremal. Let $\bar{t}_0$ be the first time such that these conditions are satisfied. We may choose ε small enough such that γ_1 (and γ_2) corresponds to a constant control , say $+1$, in $]\bar{t}_0 - \varepsilon, \bar{t}_0[$ Consider the locus:

$$\mathcal{F}(\bar{t}_0) = \{(x, \lambda) \in N_0 : \ \exists\, \varepsilon > 0, \text{ a time } t\ (\bar{t}_0 \leq t \leq \tau) \text{ and an extremal pair } (\gamma, p) \text{ such that } \gamma(t) = x, p(0) = \lambda \text{ and corresponding to constant control } +1 \text{ in }]\bar{t}_0 - \varepsilon, \bar{t}_0]; \text{ moreover the trajectory coinciding with } \gamma \text{ in the interval } [0, \bar{t}_0] \text{ and corresponding to the singular control } \varphi \text{ in } [\bar{t}_0, \bar{t}_0 + \varepsilon[\text{ is extremal}\}.$$

It is clear that (x_1, λ_1), $(x_2, \lambda_2) \in \mathcal{F}(\bar{t}_0)$ implies $\lambda_1 = \lambda_2$. From the analysis [6] it follows that generically $\mathcal{F}(\bar{t}_0)$ is a connected piecewise $\mathcal{C}^1$ manifold with boundary and it easy to check that the above assumptions implies $(x, \lambda) \in \text{int}(\mathcal{F}(\bar{t}_0))$. This conclude the proof. ■

Analyzing the boundary of $\mathcal{K}_2$ one get the following:

Theorem 4 *$N_0 \setminus \mathcal{K}_1$ is a piecewise $\mathcal{C}^1$ manifold with boundary.*

Note that if there are not overlaps (see [2]) then $\mathcal{K}_1 = \emptyset$.

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