Morse Properties for the Minimum Time Function on 2–D Manifolds

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Abstract. Given a two dimensional smooth manifold M and two smooth vector fields X and Y on M, we want to steer a point $p \in M$ to a point $q \in M$ in minimum time using only integral curves of the two vector fields X, Y. Fixing p, we define the minimum time function $\mathbf{T}_p(q)$ to reach q. We prove that, generically, $\mathbf{T}_p(q)$ is a Morse function in topological sense giving a positive answer to a question of V.I. Arnold.

Keywords: Optimal control, Morse function, Minimum time.

1 Introduction

In this paper we consider the following geometrical problem. Given a smooth manifold M and two smooth vector fields X and Y on M, we want to steer a point $p \in M$ to a point $q \in M$ in minimum time using only integral curves of the two vector fields X and Y. Even if there exists such a trajectory going from p to q, it may happen that the infimum time is obtained only by a trajectory γ whose velocity $\dot{\gamma}(t)$ belongs to the segment joining $X(\gamma(t))$ and $Y(\gamma(t))$ (not being an endpoint of it). Thus we first convexify the set of velocities letting $V(x) = \{vX(x) + (1-v)Y(x) : 0 \leq v \leq 1\}$ be the set of admissible velocities at a generic point x of M. Fixing the point p and letting q vary on M, we define the function $\mathbf{T}_p(q)$ as the minimum time to reach the point q from p. We are interested in investigating the generic properties of the function $\mathbf{T}_p$.

We treat the case in which M is two dimensional and the point p is an equilibrium point, that is $0 \in V(p)$. Even if the space of all admissible trajectories is infinite dimensional, it happens that the time optimal trajectories are indeed finite concatenations of X trajectories, Y trajectories and singular trajectories that correspond to a smooth vector field defined in terms of the Lie Bracket of X and Y. Moreover, the number of arcs is bounded on compact sets. It is known that a function defined as the minimum of a finite number of smooth functions in generic position is a Morse function in topological sense [1, 15]. We give a positive answer to the question of V.I. Arnold: is the function $\mathbf{T}_p$ generically a Morse function in topological sense? (see Theorem 1). This result is in the same spirit of [1, 15], being now the minimization taken over a finite dimensional family of functions.

In order to prove Theorem 1, we analyze the Minimum Time Fronts (briefly MTF) that are the level sets of the minimum time function $\mathbf{T}_p$. The generic properties of these fronts are illustrated in Theorem 3 of Section 2. The time optimal trajectories satisfy a first order optimality condition, the well known Pontryagin Maximum Principle (briefly PMP, see [14, 20, 23]). The trajectories satisfying the PMP are called extremals and admit a lift in the cotangent bundle that is a trajectory of a non smooth Hamiltonian. One can define the extremal front in an obvious way. It is interesting to notice that the extremal front generically develops only standard singularities as one dimensional wave front, see [2, 3]. Moreover, these singularities correspond exactly to projection singularities, of the set of lifted trajectories, of cusp type. All generic projection singularities of extremals have been classified in [6].

The generic properties of the minimum time front on points reached by abnormal extremals (that are trajectories with vanishing Hamiltonian) are studied. By a local analysis, we show that at these points the minimum time front is always tangent to the abnormal extremals, see Theorem 4. This fact was remarked also in [5] and in [24] for a quadratic cost.

The set of points q that can be reached by a trajectory starting from p in time less than or equal to $T > 0$ is called the reachable set from p in time T. The topology of this set generically changes only at a discrete set of times when a special curve of points, called overlap curve and whose points are reached by two different optimal trajectories, either starts or ends. See Theorem 5.

Finally, we study the case in which $M = \mathbb{R}^2$ and p is replaced by a smooth two dimensional manifold with boundary. In this case, under a local controllability assumption, we can also prove the semiconcavity of the minimum time function. This property is typically encountered in calculus of variation problems or optimal control problems with set of velocities $V(x)$ of the same dimension of the state space, see [12] and references therein. (Notice that, under generic conditions, the semiconcavity fails when p is a point.)

The above geometrical problem can be restated as the minimum time problem for the control system with set of velocities $V(x)$. Defining $F = \frac{Y+X}{2}$ and $G = \frac{Y-X}{2}$ this control system can be written in local coordinates as:

$$\dot{x} = F(x) + uG(x), \quad F(x_0) = 0, \quad x \in M, \quad |u| \leq 1 \tag{1}$$

where F and G are $\mathcal{C}^\infty$ vector fields on M and now x_0 plays the role of p. The properties of optimal trajectories for the control system (1), have been extensively studied in a series of papers. In [21, 22], Sussmann analyzed the properties of extremal trajectories and via a finite dimensional reduction, obtained using the PMP, proved the existence of a regular optimal synthesis for analytic systems. Roughly speaking an optimal synthesis (see [4, 11, 19]) is a map associating to each point an optimal trajectory. It happens that the synthesis is generated by a feedback which is smooth on each stratum of a Whitney stratification of the manifold. In [8, 9, 17, 18] the existence of a structurally stable optimal synthesis for generic smooth systems was proved. Moreover, a complete classification of the corresponding (non smooth) flows and relative singularities were given, in the same spirit of the work of Peixoto [16] for two dimensional dynamical systems. Finally, in [6, 7], the set of extremals in the cotangent bundle was analyzed in details and the concept of extremal synthesis was introduced.

In the following we give the definition of topological Morse function and state our main result (Theorem 1). We call respectively *reachable set within time* $T > 0$ and *reachable set* the following sets:

$$\begin{aligned} \mathcal{R}(T) \quad &:= \quad \{x \in M : \text{ there exists } t \in [0,T] \text{ and a trajectory} \\ &\qquad \gamma : [0,t] \to M \text{ of (1) such that } \gamma(0) = x_0, \ \gamma(t) = x\}. \\ \mathcal{R} \quad &:= \quad \{x \in M : \text{ there exists } t \geq 0 \text{ and a trajectory} \\ &\qquad \gamma : [0,t] \to M \text{ of (1) such that } \gamma(0) = x_0, \ \gamma(t) = x\}. \end{aligned} \tag{2}$$

Results about the structure of $\mathcal{R}$ for (1) can be found in [13], while here we are interested in the evolution of $\mathcal{R}(T)$. The minimum time function $\mathbf{T}(.) : \mathcal{R} \to \mathbb{R}^+$ (now we drop the label x_0) is by definition:

$$\mathbf{T}(x) := \inf\{t \geq 0 : \text{ there exists a trajectory } \gamma \text{ of (1) s.t. } \gamma(0) = x_0, \ \ \gamma(t) = x\}. \tag{3}$$

In general $\mathbf{T}(.)$ is not $\mathcal{C}^1$, but in our case, under generic hypotheses, it is $\mathcal{C}^\infty$ outside a rectifiable set of codimension 1. Let us introduce the following:

Definition 1 *We say that a set $Q \subset M$ is thin if $Q = \cup_{i=1}^n M_i$, with $n \in \mathbb{N} \cup \{\infty\}$, $\{M_i\}_i$ locally finite, M_i connected embedded smooth one dimensional manifold with boundary. We say that a function $f : A \subset M \to \mathbb{R}$, A open, is piecewise $\mathcal{C}^\infty$ if it is $\mathcal{C}^\infty$ except a thin set Q.*

A smooth function is said Morse function if at critical points its Hessian is non singular. To treat the case of a piecewise $\mathcal{C}^\infty$ function, we introduce the following:

Definition 2 *Let $f : A \subset M \to \mathbb{R}$ be a piecewise $\mathcal{C}^\infty$ function and $Q \subset A$ the corresponding thin set. We say that f is topologically a Morse function if, taking Q minimal, the following holds. On the set $A \backslash Q$, f is a Morse function and for each $x \in Q$ there exist a neighborhood $I(x)$ of x and an homeomorphism $\psi : I(x) \to I(x)$ such that $f \cdot \psi : I(x) \to \mathbb{R}$ is a Morse function.*

Thus a topologically Morse function is a function whose level sets are homeomorphic to one of the cases of the following figure:

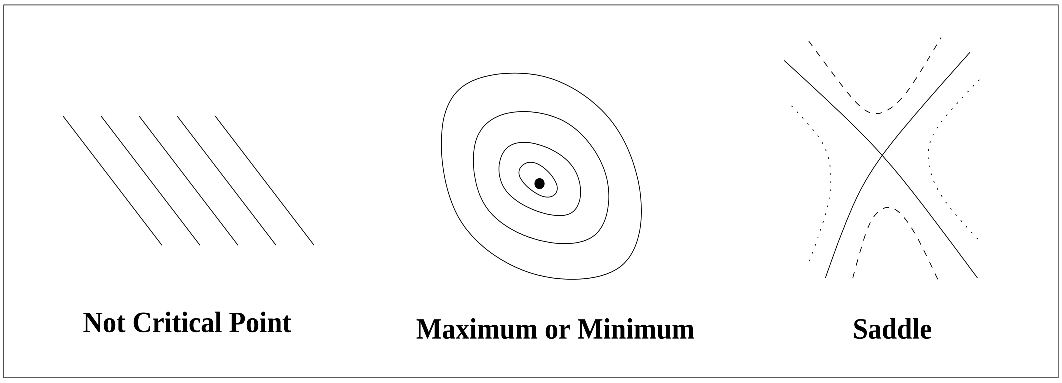


Figure 1.1: Level sets of a topologically Morse function

Theorem 1 **(Morse Property)** *Fix $\mathcal{T} > 0$. Under generic conditions on F and G, on the interior of the set $\mathcal{R}(\mathcal{T})$, the minimum time function $\mathbf{T}(.)$ is a piecewise $\mathcal{C}^\infty$ function and topologically a Morse function.*

In Section 2 we recall some basic tools of [6, 7, 8, 17, 18]. In Section 3 we introduce the definition of *optimal strip*, similar to the one of *strip* used in [6], and we study the local properties of the MTF. In Section 4 we prove that the minimum time function is topologically a Morse function. Finally in Section 5 we discuss the case of a two-dimensional initial manifold on the plane.

2 Basic Definitions

Definition 3 *We say that a vector field is $\mathcal{C}^\infty$ if its components admit partial derivatives of any order that are bounded on the whole manifold.*

Let Ξ be the set of all couples of $\mathcal{C}^\infty$ vector fields (F, G) such that $x_0 \in M$ is an equilibrium point for F, that is $F(x_0) = 0$. From now on we endow Ξ with the $\mathcal{C}^3$ topology, that is the topology induced by the norm:

$$\begin{aligned} \|(F,G)\| &= \sup\Big\{ \left|\frac{\partial^{\alpha_1+\alpha_2} F_i(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right|, \left|\frac{\partial^{\alpha_1+\alpha_2} G_i(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right| : x \in M; \\ & \qquad i = 1,2;\ \alpha_1, \alpha_2 \in \mathbf{N} \cup \{0\}; \alpha_1 + \alpha_2 \le 3 \Big\}. \end{aligned}$$

We say that a subset of Ξ is generic if it contains an open and dense set. Analogously a property P is said generic if the set of points satisfying P is generic.

We fix once for all $\mathcal{T} > 0$ and for every $(F, G) \in \Xi$ we consider the problem of reaching every point of $\mathcal{R}(\mathcal{T})$ in minimum time from x_0 for the control system (1).

Given a measurable function $u : [a, b] \to [-1, 1]$, a trajectory of (1) corresponding to u is an absolutely continuous map $\gamma : [a, b] \to M$ such that $\dot\gamma(t) = F(\gamma(t)) + u(t)G(\gamma(t))$ for almost every $t \in [a, b]$. Since the system is autonomous we can always assume that $Dom(\gamma) = [0, a]$ for some $a \in \mathbb{R}$, where Dom denotes the domain. A trajectory $\gamma : [0, a] \to M$ is (time) optimal if for every trajectory $\gamma' : [0, b] \to M$ with $\gamma(0) = \gamma'(0)$ and $\gamma(a) = \gamma'(b)$ we have $a \le b$. A trajectory γ corresponding to a constant control ± 1 is called a *bang arc*. A *bang-bang* trajectory is a finite concatenation of bang arcs and a time where the control changes sign is called *switching time*. In this special case, the well known Pontryagin Maximum Principle states the following. Define for every $(x, p, u) \in T^*M \times [-1, 1]$, where T^*M is the cotangent bundle of M, the following:

$$\mathcal{H}(x, p, u) = < p, F(x) > +u < p, G(x) > \ \in \ \mathbb{R}, \tag{4}$$

and:

$$H(x, p) = \max\{\mathcal{H}(x, p, u) : u \in [-1, 1]\}. \tag{5}$$

Theorem 2 (PMP) *If $\gamma : [0,a] \to M$ is a (time) optimal trajectory corresponding to a control $u : [0,a] \to [-1,1]$, then there exist a non trivial field of covectors along γ, that is a function $\lambda : t \in [0,a] \mapsto \lambda(t) \in T^*_{\gamma(t)}M$ never vanishing, and a constant $\lambda_0 \leq 0$ such that*

i) $\dot{\lambda}(t) = - < \lambda(t), (\nabla F + u(t)\nabla G)(\gamma(t)) >$,

ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$ *for almost every* $t \in Dom(\gamma)$,

iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$ *for almost every* $t \in Dom(\gamma)$.

Notice that the PMP provides a condition that is only necessary for optimality. Every pair (γ, λ) satisfying this condition is called extremal. Moreover if γ is optimal we say that the pair (γ, λ) is optimal.

Definition 4 *If (γ, λ) is an extremal pair and the corresponding Hamiltonian satisfies $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0$ for almost every $t \in Dom(\gamma)$ (i.e. if λ_0 in ii) above vanishes) then we say that (γ, λ) is an abnormal extremal.*

Using the PMP, it is possible to show that the extremal trajectories have a special structure. Let us introduce some notation to explicit this fact.

Recall $Y = F + G$ and $X = F - G$. Let $\gamma : [t_1, t_2] \to M$ be a trajectory of (1). If γ corresponds to constant control +1 (resp. -1) in $[t_1, t_2]$ we say that $\gamma|_{[t_1,t_2]}$ is a Y–trajectory (resp. X–trajectory) and we write $\gamma|_{[t_1,t_2]} \in$ Traj(Y) (resp. $\gamma|_{[t_1,t_2]} \in$ Traj(X)). We say that $\gamma|_{[t_1,t_2]}$ is a Z–trajectory and we write $\gamma|_{[t_1,t_2]} \in$Traj(Z), if γ in $[t_1, t_2]$ corresponds to the control (called singular):

$$\varphi(x) = -\frac{< \nabla\Delta_B(x), F(x) >}{< \nabla\Delta_B(x), G(x) >},$$

where:

$$\Delta_B(x) = G(x) \wedge [F,G](x) = G_1(x)[F,G]_2(x) - G_2(x)[F,G]_1(x),$$

and $[F,G]$ is the Lie bracket of F and G. Moreover if $\gamma_1 : [t_1, t_2] \to M$, $\gamma_2 : [t_2, t_3] \to M$ $(t_1 < t_2 < t_3)$ are two trajectories of (1) with $\gamma_1(t_2) = \gamma_2(t_2)$, we set $(\gamma_2 * \gamma_1)(t) := \gamma_1(t)$ if $t \in\ [t_1, t_2[$ and $(\gamma_2 * \gamma_1)(t) := \gamma_2(t)$ if $t \in\ [t_2, t_3]$. In [17] it was shown that under generic conditions (see (P1)–(P9) in [6]) each extremal trajectory γ can always be written in the form $\gamma = \gamma_{n(\gamma)} * ... * \gamma_1$ with $n(\gamma) \in \mathbb{N}$ where each arc γ_i is bang or singular (that is a Z–trajectory). The function $n(\gamma)$ is called the number of arcs of γ. For later use we also define the function:

$$\Delta_A(x) = F(x) \wedge G(x) = F_1(x)G_2(x) - F_2(x)G_1(x).$$

On the set of zeros of Δ_A, F and G are parallel. Moreover (see [7]) all the switchings of abnormal extremals occur exactly on this set.

Under generic hypotheses, (see [17]), an optimal trajectory $\gamma \in$ Traj(Z) iff $Supp(\gamma) \subset \Delta_B^{-1}(0)$. In this case we say that $Supp(\gamma)$ is a *turnpike*. Moreover every extremal trajectory exits x_0 with constant control +1 or -1.

We call $\gamma^\pm : [0, t_f^\pm] \to M$ the extremal trajectories originating from x_0 and corresponding to a constant control ± 1, where $t_f^\pm$ are either the last times in which $\gamma^\pm$ are extremal or $t_f^\pm = \mathcal{T}$. Notice that if $t_{op}^\pm$ are the last times in which $\gamma^\pm$ are optimal, we have $t_{op}^\pm \leq t_f^\pm$. We define $\gamma_{op}^\pm = \gamma^\pm|_{[0,t_{op}^\pm]}$.

Definition 5 *Let (γ, λ) be an extremal pair. The corresponding switching function is defined as $\phi(t) :=< \lambda(t), G(\gamma(t)) >$.*

We have the following:

Proposition 1 *Let (γ, λ) be an extremal pair and let t_1, t_2 be two times such that $(t_1 < t_2)$. Then:*

- *on $[t_1, t_2]$, γ corresponds to constant control +1 (resp. -1) iff for each $t \in [t_1, t_2]$ we have $\phi(t) \geq 0$ (resp. $\phi(t) \leq 0$) and meas$(\{t \in [t_1, t_2] : \ \phi(t) = 0\}) = 0$;*
- *on $[t_1, t_2]$, γ corresponds to the singular control φ iff for each $t \in [t_1, t_2]$ we have $\phi(t) = 0$.*

Definition 6 *A synthesis for the control system* $\Sigma \in \Xi$ *is a family* $\Gamma = \{\gamma_x : [0, b_x] \mapsto M,\ x \in \mathcal{R}(\mathcal{T})\}$ *of trajectories satisfying the following conditions:*

- *for each* $x \in \mathcal{R}(\mathcal{T})$ *one has* $\gamma_x(0) = x_0$, $\gamma_x(b_x) = x$;
- *if* $y = \gamma_x(t)$, *where* $t \in Dom(\gamma_x)$, *then* $\gamma_y = \gamma_x|_{[0,t]}$.

A synthesis for the system Σ *is time optimal if, for each* $x \in \mathcal{R}(\mathcal{T})$ *one has* $\gamma_x(\mathbf{T}(x)) = x$, *i.e.* $b_x = \mathbf{T}(x)$, *where* $\mathbf{T}(.)$ *is the minimum time function defined in (3).*

Under generic assumptions, the problem of reaching in minimum time every point of the reachable set for the system (1) admits a regular synthesis. Moreover there exists a stratification of $\mathcal{R}(\mathcal{T})$ such that the optimal synthesis can be obtained from a feedback $u(x)$ satisfying:

- on strata of dimension 2, we have $u(x) = \pm 1$,
- on strata of dimension 1, called frame curves (in the following FCs), $u(x) = \pm 1$ or $u(x) = \varphi(x)$.

The strata of dimension 0 are called frame points (in the following FPs). Every FP is an intersection of two FCs. A FP x that is the intersection of two frame curves F_1 and F_2 is called a (F_1, F_2) Frame Point. In [18] it is provided a complete classification of all types of FPs and FCs, under generic conditions. All the possible FCs are the following:

- FCs of kind Y (resp. X), that correspond to subsets of γ^+_{op} (resp. subsets of γ^-_{op}),
- FCs of kind C, called *switching curves*, that are curves made of switching points,
- FCs of kind S, i.e. turnpikes,
- FCs of kind K, called overlaps, reached optimally by two trajectories coming from different directions.
- FCs that are arcs of optimal trajectories that start at FPs. These trajectories "transport" special information. Here we need to distinguish three kind of these curves:
  - curves of kind γ_A that are abnormal extremals,
  - curves of kind γ_k that are curves starting at the terminal point of an overlap (i.e. that start at the FPs of kind $(Y, K)_3$ $(S, K)_2, (C, K)_1$, see below)
  - curves of kind γ_0 that are the other arcs of optimal trajectories that start at FPs.

There are seventeen topological equivalence classes of FPs: (X, Y), $(Y, C)_{1,2,3}$, (Y, S), $(Y, K)_{1,2,3}$, $(C, C)_{1,2}$, $(C, S)_{1,2}$, $(C, K)_{1,2}$, $(S, K)_{1,2}$, (K, K). The optimal synthesis near each Frame Point is showed in Figure 2.1.

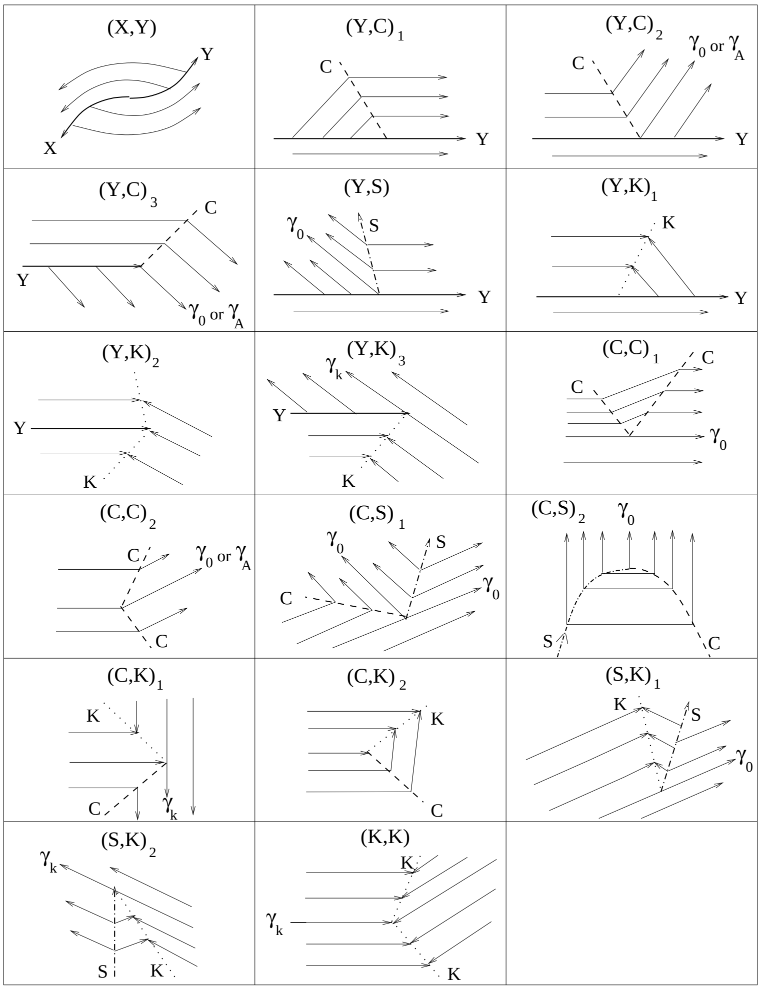
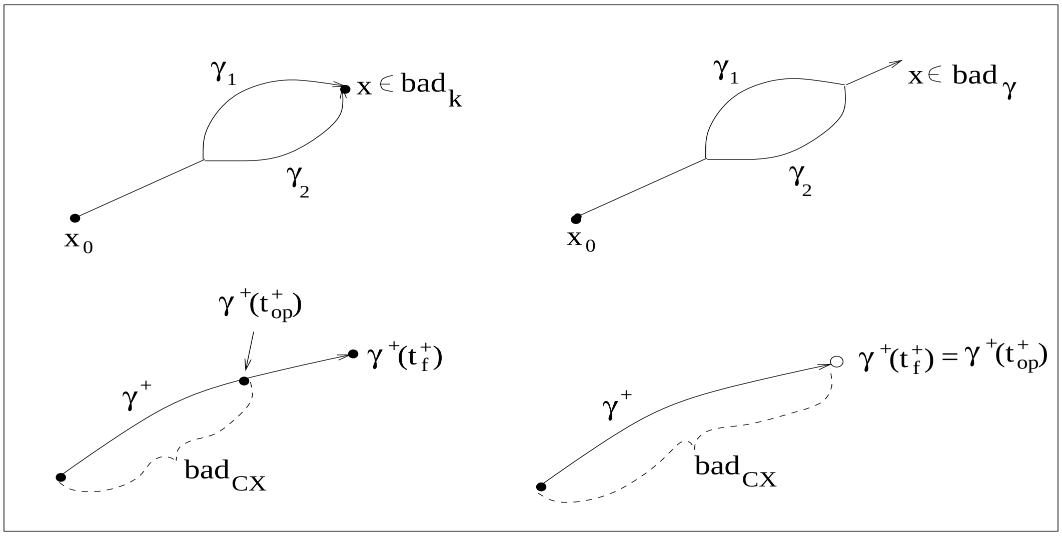


Figure 2.1: the FPs of the Optimal Synthesis.

Definition 7 *We define the following sets of bad points:*

- bad_k := $\{x \in \mathcal{R}(\mathcal{T})$: *there exist two optimal trajectories* $\gamma_1,\ \gamma_2 : [0,T] \to M$*,* $T > 0$ *and* $\varepsilon > 0$ *such that* $\gamma_1(0) = \gamma_2(0) = x_0$*,* $\gamma_1(T) = \gamma_2(T) = x$*,* $\gamma_1(t) \neq \gamma_2(t)$ *for every* $t \in [T-\varepsilon, T[\}$.
- bad_γ := $\{x \in \mathcal{R}(\mathcal{T})$: *there exist two optimal trajectories* $\gamma_1,\ \gamma_2 : [0,T] \to M$*,* $T > 0$*,* $a \in]0,T[$ *and* $0 < \varepsilon < a$ *such that* $\gamma_1(0) = \gamma_2(0) = x_0$*,* $\gamma_1(T) = \gamma_2(T) = x$*,* $\gamma_1(t) = \gamma_2(t)$ *for* $t \in [a,T]$ *and* $\gamma_1(t) \neq \gamma_2(t)$ *for* $t \in [a-\varepsilon, a[\}$.
- $bad_{CV} := bad_k \cup bad_\gamma$.
- $bad_{CX} := Supp(\gamma^+_{op}) \cup Supp(\gamma^-_{op}) \setminus (\gamma^+(t^+_f) \cup \gamma^-(t^-_f))$

Remark. Of course bad_k is the set of all the overlap FCs and bad_γ is the set of γ_k's. Moreover the set bad_{CX} is the optimal part of $Supp(\gamma^\pm)$ if $t^\pm_{op} < t^\pm_f$, otherwise it is $Supp(\gamma^\pm)$ without the terminal point. See the following figure:



The minimum time front in time $0 < T < \mathcal{T}$ (in the following MTF) is the T–level surface of $\mathbf{T}(x)$:

$$F_T = \{x \in \mathcal{R}(\mathcal{T}) : \quad \mathbf{T}(x) = T\}. \tag{6}$$

We want to prove that, under generic conditions on (F, G) and for generic T, the corresponding MTF is a one dimensional piecewise $\mathcal{C}^1$ compact manifold. More precisely the result is valid for every T such that the following conditions hold.

(C1) No FP of kind $(Y, K)_2$, $(Y, K)_3$ or (K, K) is on the front F_T ,

(C2) The front F_T is not tangent to bad_k.

Definition 8 *Let $x \in F_T$. We define the Boltianskii contingent cone to the set $\mathcal{R}(\mathbf{T}(x))$ at x as:*

$$C^C(\mathcal{R}(\mathbf{T}(x)), x) = \{v \in T_xM : \exists\ \xi :]-\varepsilon, \varepsilon[\to \mathcal{R}(\mathbf{T}(x)) \text{ smooth } s.t.\ \xi(0) = x, \dot{\xi}(0) = v\}$$

and the normal cone as:

$$C^N(\mathcal{R}(\mathbf{T}(x)), x) = \{w \in T^*_xM : \quad < w, v > \leq 0, \textit{ for all } v \in C^C(\mathcal{R}(\mathbf{T}(x)), x)\}$$

Let $x \in \partial\mathcal{R}(\mathbf{T}(x))$. Notice that if $\partial\mathcal{R}(\mathbf{T}(x))$ is smooth at x then $C^N(\mathcal{R}(\mathbf{T}(x)), x) = \{\alpha\nabla\mathbf{T}(x) : \ \alpha \geq 0\}$. When $M = \mathbb{R}^2$ and $\partial\mathcal{R}(\mathbf{T}(x))$ has a corner at x, then if $\mathcal{R}$ is convex at x then C^N is a cone with nonempty interior and if $\mathcal{R}$ is concave then $C^N = \{0\}$. In the next session, we will prove the following:

Theorem 3 (Smoothness Properties of F_T) *Under generic conditions on (F, G), for $0 < T < \mathcal{T}$ satisfying conditions* **(C1)** *and* **(C2)***, F_T is a one–dimensional piecewise $\mathcal{C}^1$ compact embedded submanifold of M. Moreover, $x \in F_T$ is a point at which F_T is not $\mathcal{C}^1$ iff:*

(case 1) *$x \in bad_{CV}$ or*

(case 2) *$x \in bad_{CX}$;*

For every such x we have $C^N(\mathcal{R}(T), x) = \{0\}$ (case 1) or $Int(C^N(\mathcal{R}(T), x)) \neq \emptyset$ (case 2), i.e. if $M = \mathbb{R}^2$ then $\mathcal{R}(T)$ is concave at x if $x \in bad_{CV}$ and convex if $x \in bad_{CX}$.

Theorem 4 (F_T on Abnormal Extremals) *Fix $0 < T < \mathcal{T}$, let $\bar{x} \in F_T$ and suppose $\bar{x} \notin bad_{CV} \cup bad_{CX}$. Let $\bar{\gamma}$ be the unique optimal trajectory reaching $\bar{x}$. Then, under generic conditions, F_T is tangent to $Supp(\bar{\gamma})$ at $\bar{x}$ iff $\bar{\gamma}$ is an abnormal extremal.*

Moreover, from the analysis of [9, 18], it easily follows:

Theorem 5 (Topological Properties of of $\mathcal{R}(T)$) *For every T, $\mathcal{R}(T)$ is connected. Moreover, if bad_k is empty, $\mathcal{R}(T)$ is simply connected. At the times T_0 when* **(C1)** *or* **(C2)** *fails, the topology of the reachable set changes. In the planar case, there exists $n \in \mathbb{N}$ such that for t in a left neighborhood of T_0 the reachable set $\mathcal{R}(t)$ is homeomorphic to a disc with n holes and for t in a right neighborhood of T_0, $\mathcal{R}(t)$ is homeomorphic to a disc with $n+1$ holes or $n-1$ holes. For a general manifold it may also happen that we add a handle.*

Notice that if the first part of Theorem 5 applies, that is $bad_k = \emptyset$, then the minimum time function has no critical point except x_0 and thus the level sets are homeomorphic to circles. More precisely, outside $Supp(\gamma^{\pm}_{op})$ the level sets are diffeomorphic to circles. In this case, trivially one concludes that the minimum time function topologically is a Morse function.

3 Properties of the Minimum Time Front

To prove Theorems 3 and 4, we first study the smoothness properties of the MTF in a neighborhood of $Supp(\gamma^{+}_{op}) \cup Supp(\gamma^{-}_{op})$. From now on, for simplicity, we consider the case $M = \mathbb{R}^2$ and $x_0 = 0$. The conclusions are valid in the general case mutatis mutandis.

3.1 The Optimal Syntheses in a Neighborhood of $\gamma^{\pm}_{op}$

The shape of the optimal synthesis in a neighborhood of $Supp(\gamma^{+}_{op}) \cup Supp(\gamma^{-}_{op})$ was described in details in [8]. Here we recall the results we need. Given an extremal trajectory γ, let us define $v^{\gamma}(v_0, t_0; t)$ to be the solution to the Cauchy problem:

$$\left\{ \begin{array}{lcl} \dot{v^{\gamma}}(v_0, t_0; t) & = & (\nabla F + u(t)\nabla G)(\gamma(t)) \cdot v^{\gamma}(v_0, t_0; t) \\ v^{\gamma}(v_0, t_0; t_0) & = & v_0, \end{array} \right. \tag{7}$$

where $u(.)$ is the control corresponding to γ. In [17], denoting $\bar{v}^{\gamma}(t) := v^{\gamma}(G(\gamma(t)), t; 0)$ it was defined the function $\theta^{\gamma} : Dom(\gamma) \to [-\pi, \pi]$, $\theta^{\gamma}(t) := \arg(\bar{v}^{\gamma}(0), \bar{v}^{\gamma}(t))$, where arg is the angle measured counterclockwise. In the following we use the notation $\theta^{\pm}(t) := \theta^{\gamma^{\pm}}(t)$.

In [8, 17] it was shown:

Proposition 2 *We have the following:*

(A) $t^{+}_{f} = \left\{ \begin{array}{l} \mathcal{T} \textit{ if } |\theta^{+}(a) - \theta^{+}(b)| < \pi \ \ \forall \ \ a,\ b \in \ [0, \mathcal{T}]; \\ \min\{t \in [0, \mathcal{T}] : \ \ |\theta^{+}(a) - \theta^{+}(b)| = \pi \textit{ for some } a,\ b \in \ [0, t]\} \textit{ otherwise}; \end{array} \right.$

(B) *let γ be an extremal trajectory, then for almost every $t \in Dom(\gamma)$ we have $sgn(\dot{\theta}^{\gamma}(t)) = sgn(\Delta_B(\gamma(t))$;*

(C) *let γ be an extremal trajectory, then for each $t \in Dom(\gamma)$ the conditions $\Phi(t) = \lambda(t) \cdot G(\gamma(t)) = 0$, $\dot{\theta}^{\gamma}(t) \neq 0$ imply that t is a switching time for γ.*

Moreover the following times were single out in the interval $[0, t^{\pm}_{op}]$:

Definition 9

$$\begin{aligned} t^{+}_{0} &= t'^{+}_{0} = 0 \\ t^{+}_{1} &= \inf\{t > 0 : \ \ \theta^{+} \textit{ has a local max. at } t, \ \ \theta^{+}(t) > 0\} \\ t^{+}_{i} &= \inf\{t > t^{+}_{i-1} : \ \ \theta^{+} \textit{ has a local max. at } t, \ \ \theta^{+}(t) > \theta^{+}(t^{+}_{i-1})\} \\ t'^{+}_{1} &= \inf\{t > 0 : \ \ \theta^{+} \textit{ has a local min. at } t, \ \ \theta^{+}(t) < 0\} \\ t'^{+}_{i} &= \inf\{t > t'^{+}_{i-1} : \ \ \theta^{+} \textit{ has a local min. at } t, \ \ \theta^{+}(t) < \theta^{+}(t^{+}_{i-1})\} \end{aligned}$$

$$s_i^+ = \begin{cases} \max\{t \in [t_{i-1}^+, t_i^+]: \quad \theta^+(t) = \theta^+(t_{i-1}^+)\} \textit{ if } t_i^+ \textit{ is defined,} \\ \max\{t \in [t_{i-1}^+, t_{op}^+]: \quad \theta^+(t) = \theta^+(t_{i-1}^+)\} \textit{ otherwise} \end{cases}$$

$$s_i'^+ = \begin{cases} \max\{t \in [t_{i-1}'^+, t_i'^+]: \quad \theta^+(t) = \theta^+(t_{i-1}'^+)\} \textit{ if } t_i'^+ \textit{ is defined,} \\ \max\{t \in [t_{i-1}'^+, t_{op}'^+]: \quad \theta^+(t) = \theta^+(t_{i-1}'^+)\} \textit{ otherwise.} \end{cases}$$

Similarly the times t_i^-, $t_i'^-$, s_i^-, $s_i'^-$, were defined.

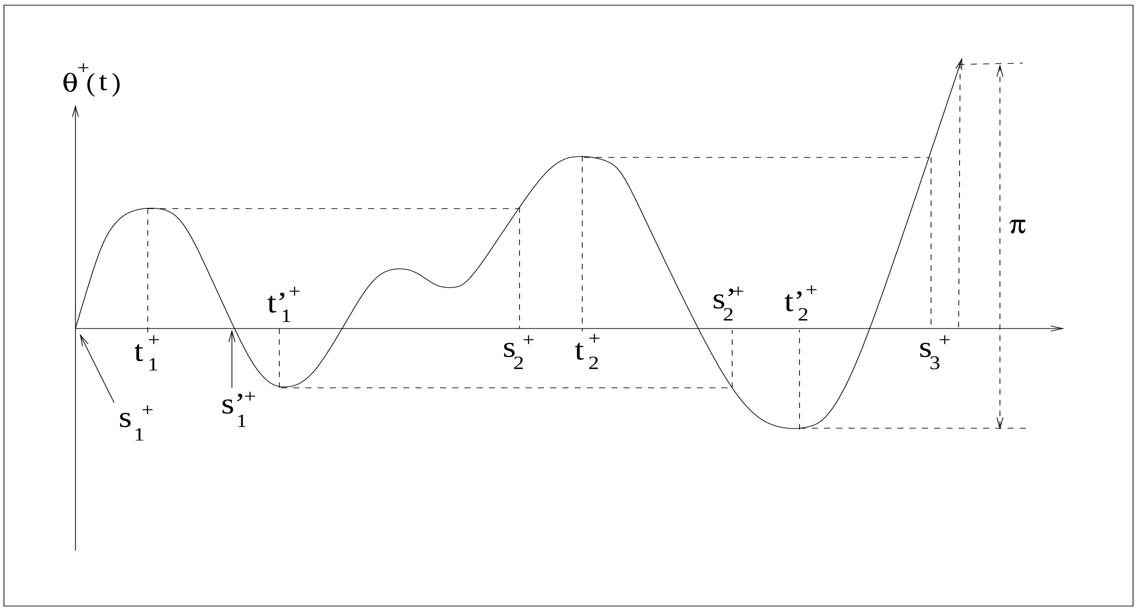


Figure 3.1: An example of function $\theta^+(t)$ in the case $t_{op}^+ = t_f^+ < \mathcal{T}$.

Under generic conditions, the structure of the optimal synthesis in a neighborhood of $Supp(\gamma_{op}^+) \cup Supp(\gamma_{op}^-)$ is illustrated by the following Theorem that includes Propositions 3.1, 3.2, 3.3 of [8] (see [8] for the explicit statement of the generic conditions):

Theorem 6 *Under generic conditions:*

- *if t^* lies in one of the open intervals $]s_i^+, t_i^+[$ or $]s_i'^+,\ t_i'^+[$ then there exists an extremal control of the form:*

$$u(t) = \begin{cases} 1 & \textit{if} \quad t \in [0, t^*[, \\ -1 & \textit{if} \quad t \in [t^*, t^* + \varepsilon[, \end{cases} \tag{8}$$

  for some $\varepsilon > 0$. In the first case the trajectory bifurcates to the right of γ_{op}^+ (clockwise), in the second case to the left (counterclockwise). On the other hand, no extremal control of the form (8) exists if t^* is not contained in any one of the closed intervals $[s_i^+, t_i^+]$ or $[s_i'^+,\ t_i'^+]$;

- *to the right of each point $\gamma_{op}^+(t_i^+)$ ($t_i^+ > 0$) and to the left of each point $\gamma_{op}^+(t_i'^+)$ ($t_i'^+ > 0$) it originates either a turnpike, when the inner products $\nabla\Delta_B \cdot X$ and $\nabla\Delta_B \cdot Y$ have opposite signs, that is when the two vector fields X, Y points to opposite sides of the set of zeros of Δ_B, or a switching curve when the signs are equal;*

- *to the right of each point $\gamma_{op}^+(s_i^+)$ ($i \geq 2$) and to the left of each point $\gamma_{op}^+(s_i'^+)$ ($i \geq 2$) the extremal synthesis contains either a switching curve C or an overlap curve K;*

- *if $s_1^+ > 0$ (resp. $s_1'^+ > 0$) then to the right of $\gamma_{op}^+(s_1^+)$ (resp. to the left of $\gamma_{op}^+(s_1'^+)$) the extremal synthesis contains either a switching curve C or an overlap curve K.*

- *if* $t^+_{op} = t^+_f < \mathcal{T}$ *then a switching curve originates from* $\gamma^+_{op}(t^+_{op})$. *Such curve bifurcates to the left of* γ^+_{op} *when* $\dot\theta^+(t^+_{op}) > 0$, *and to the right of* γ^+_{op} *when* $\dot\theta^+(t^+_{op}) < 0$;
- *if* $t^+_{op} < t^+_f$ *and* $t^+_{op} \in]s^+_i, t^+_i[$ *or* $t^+_{op} \in]s'^+_i,\ t'^+_i[$, *then an overlap curve originates from* $\gamma^+_{op}(t^+_{op})$. *Such curve bifurcates to the left of* γ^+_{op} *when* $\dot\theta^+(t^+_{op}) > 0$, *and to the right of* γ^+_{op} *when* $\dot\theta^+(t^+_{op}) < 0$;
- *if* $t^+_{op} < t^+_f$ *and* $t^+_{op} \notin \bigcup_i([s^+_i, t^+_i] \cup [s'^+_i,\ t'^+_i])$, *then two overlap curves bifurcate from* $\gamma^+_{op}(t^+_{op})$.

A similar result holds for γ^-_{op} under similar generic assumptions.

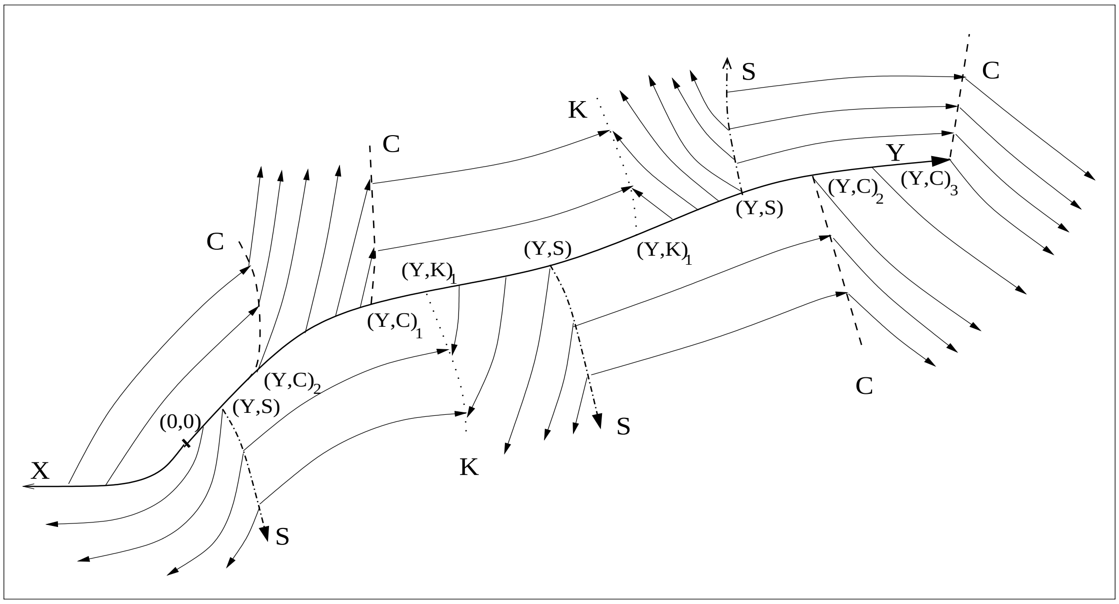


Figure 3.2: An example of optimal synthesis corresponding to the function $\theta^+(t)$ of Figure 3.1.

3.2 Definition of Strips

Now we give the key definition of *optimal strip*. An optimal strip is essentially a one parameter continuous family of optimal trajectories formed by the same sequence of arcs.

Definition 10 *Let* a, b *be two real numbers s.t.* $0 \le a < b \le \mathcal{T}$, $x \in \mathcal{R}(\mathcal{T})$ *and* $f : [a, b] \to \mathbb{R}$ *a function such that* $f(\alpha) \ge \alpha$. *A set of trajectories* $\mathcal{S}^{a,b,x,f} = \{\gamma_\alpha : [0, f(\alpha)] \to M,\ \alpha \in [a, b],\ \gamma_a(a) = x\}$ *is called an optimal strip if:*

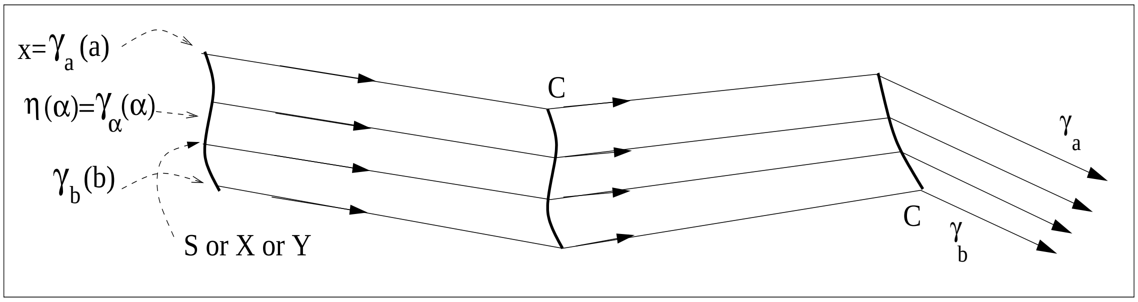
i) $\forall\ \alpha \in [a, b]$, $\gamma_\alpha : [0, f(\alpha)] \to M$, *is an optimal trajectory for the control problem (1). Moreover there exists* $\varepsilon > 0$ *such that* $\gamma_{[\alpha,\alpha+\varepsilon]}$ *corresponds to a constant control* ± 1;

ii) $\forall\ \alpha \in]a, b[$, γ_α *does not switch on* $\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$ *after time* α;

iii) *The set* $\mathcal{B}^{a,b,x,f} = \{y \in \mathcal{R} :\ \exists\ \alpha \in\ ]a, b[$ *and* $t \in]\alpha, f(\alpha)[$ *such that* $y = \gamma_\alpha(t)$, t *is a switching time for* $\gamma_\alpha\}$ *is never tangent to* X *or* Y.

iv) *The map* $\eta :\ \alpha \in\ [a, b] \mapsto \gamma_\alpha(\alpha) \in M$ *is a bang or singular arc and for* $a \le \alpha' \le \alpha \le b$, *it holds* $\gamma_\alpha(t) = \gamma_{\alpha'}(t)$ *for* $t \in [0, \alpha']$.

The function η is called the *base* of the optimal strip, moreover $Int(\mathcal{S}^{a,b,x,f}) := \{\gamma_\alpha :\ \alpha \in]a, b[\}$ is called an *open optimal strip* and $\partial\mathcal{S}^{a,b,x,f} := \{\gamma_a,\ \gamma_b\}$ is called *optimal strip border.* The concept of strip is clarified in the following picture:



The construction of the optimal synthesis, as in [8], can be done in the following way. One first constructs all optimal strips bifurcating from $\gamma_{op}^{\pm}$. Then one studies the evolution of each strip. It can happen that an optimal strip is divided into two strips when some trajectory of the strip enters a turnpike. This happens at Frame Points of kind $(S,C)_1$ and $(S,K)_1$ (moreover two strips with the turnpike as base start at these points). A strip can terminate on some K curve. Finally, a strip can glue together with another strip at Frame Points of kind $(Y,K)_3$, $(C,K)_1$ and $(S,K)_2$ (indeed the strip is divided into two parts one of which ends on the K curve and the other glues together with another strip).
Let $\{S_M^i\}_{i\in I}$ be the set of all the maximal strips i.e. the strips with maximal base and maximal time $f(\alpha)$. Clearly we may split the synthesis Γ as:

$$\Gamma = \bigcup_{i\in I} S_M^i.$$

This partition of the optimal synthesis can be realized with an algorithm entirely similar to the one of [6] and permits to study the evolution of the MTF and the properties of the minimum time function separately for the base, the borders, the internal of the strips and for the overlap curves. Notice that if $S^{a,b,x,f}$ is a maximal strip then $f(\alpha) < \mathcal{T}$ for some $\alpha \in]a,b[$ iff there exists an overlap curve K such that $\gamma_\alpha(f(\alpha)) \in K$. We can split the maximal strips in such a way that they satisfy the following:

v) either $\{\gamma_\alpha(f(\alpha)), \alpha \in [a,b]\}$ is a K curve or $f(\alpha) = \mathcal{T}$ for every $\alpha \in [a,b]$.

Moreover notice that every FC of the optimal synthesis, that is not an overlap curve, is the support of a base or the support of a border of an optimal strip.

In subsection 3.3 we study the evolution of F_T in a neighborhood of $x_0 = 0$, while in subsection 3.4 in a neighborhood of $\gamma_{op}^+ \cup \gamma_{op}^-$. Then in subsection 3.5 we study the evolution of the MTF on the other bases of the strips (i.e the turnpikes), in subsection 3.6 on the internal of the strips, in subsection 3.7 on overlap curves and finally in subsection 3.8 on the borders.

3.3 F_T in a Neighborhood of the Origin

Choose a local system of coordinates such that:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(x) = \begin{pmatrix} -1 + O(x_1, x_2) \\ -b\, x_1 + b' x_2 + O(|x|^2) \end{pmatrix}, \quad b < 0. \tag{9}$$

The expressions for γ_{op}^+ and γ_{op}^- are:

$$\gamma_{op}^+(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \gamma_{op}^-(t) = \begin{pmatrix} -t + O(t^2) \\ \frac{1}{2}bt^2 + O(t^3) \end{pmatrix}. \tag{10}$$

Let us compute the MTF along γ_{op}^+, in the region where $x_2 < 0$. Consider a trajectory γ^α corresponding to constant control -1 on the interval $[0,\alpha[$ and to constant control +1 on the interval $[\alpha, T]$ where α belongs to the interval $[0,T]$. Using the expressions (10) it follows that $\gamma^\alpha(T)$ has coordinates $x_1(\alpha) = T - 2\alpha + O(\alpha^2)$, $x_2(\alpha) = \frac{1}{2}b\alpha^2 + O(\alpha^3)$, from which it follows the expression for the MTF:

$$x_2 = \frac{1}{8}b(x_1 - T)^2 + O(x_1^3). \tag{11}$$

Let us compute the MTF along γ_{op}^+, in the region where $x_2 > 0$. A point with $x_2 > 0$ is reached at time T by a trajectory corresponding to constant control +1 in the interval $[0, T-\alpha[$ and to constant control -1 in the interval $[T-\alpha, T]$ where α belong to the interval $[0, T]$. Consider now an optimal trajectory γ_η^- of (1), corresponding to constant control -1, and having initial condition $\gamma_\eta^-(0) = (\eta, 0)$. We have:

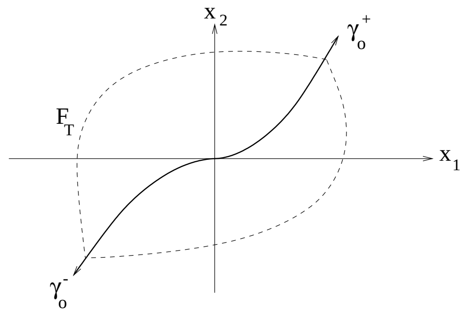
$$\gamma_\eta^-(t) = \begin{pmatrix} \eta - t + O(t^2, t\eta) \\ \frac{1}{2}bt^2 - b\eta t + O(t^3, \eta^2 t, \eta t^2) \end{pmatrix}. \tag{12}$$

Similarly to the previous case we obtain the expression for the MTF:

$$x_2 = \frac{1}{2}bT(x_1 - T) + \frac{3}{8}b(x_1 - T)^2 + O((x_1 - T)^2).$$

In the same way we may compute the MTF along γ_{op}^-, moreover the MTF is clearly smooth on $I(0) \setminus Supp(\gamma_{op}^+) \cup Supp(\gamma_{op}^-)$, where $I(0)$ is a neighborhood of the origin. This results can be summarized in the following (see figure):

Proposition 3 *For $T > 0$, sufficiently small, F_T is a 1–dimensional piecewise $\mathcal{C}^1$ embedded compact submanifold of M. The only two points in which F_T is not $\mathcal{C}^1$ are the points $F_T \cap Supp(\gamma_{op}^+)$ and $F_T \cap Supp(\gamma_{op}^-)$. Moreover if x is a such point we have $Int(C^N(\mathcal{R}(T), x)) \neq 0$, that is $\mathcal{R}(T)$ is convex at x.*



3.4 F_T in a Neighborhood of $\gamma_{op}^+ \cup \gamma_{op}^-$

Let $I(\gamma_{op}^\pm)$ be a neighborhood of $Supp(\gamma_{op}^+) \cup Supp(\gamma_{op}^-)$. In this subsection we study the smoothness properties of $F_T \cap I(\gamma_{op}^\pm)$. For this purpose we analyze F_T in a neighborhood of all the $(Y, \ .\ )$ FPs ordered by increasing time.

Analysis up to the first (YS) or $(YC)_1$ FP.

Assume $t_1'^+ > 0$ (being the case $t_1^+ > 0$ entirely similar) and suppose that at the point $\gamma_{op}^+(t_1'^+)$ a turnpike starts. In this case it bifurcates on the left of γ_{op}^+. Choose a system of coordinates such that:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(x) = \begin{pmatrix} a_0 + a_1(x_1 - t_1'^+) + a_2 x_2 + O((x_1 - t_1'^+)^2, x_2^2) \\ b_0 + b_1(x_1 - t_1'^+) + b_2 x_2 + O((x_1 - t_1'^+)^2, x_2^2) \end{pmatrix}, \quad a_0 < 1, \quad b_0 > 0. \tag{13}$$

To have a (Y, S) point at $(t_0', 0)$ the following conditions must hold:

1) $(t_1'^+, 0) \in \Delta_B^{-1}(0)$ that implies:

$$a_2 = \frac{a_1 b_0}{a_0 - 1}.$$

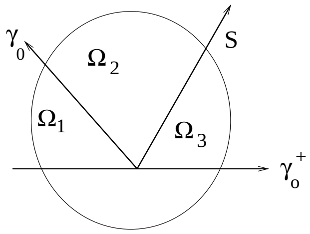
Under this condition, the set $\Delta_B^{-1}(0)$ is locally described by the equation:

$$x_2 = m_s(x_1 - t_1'^+) + O((x_1 - t_1'^+)^2) \text{ where } m_s = \frac{a_1 a_2 - a_1 b_1}{a_2^2 - a_1 b_2},$$

and we assume the generic conditions $a_2^2 - a_1 b_2 \neq 0$, $m_s \neq 0$. The case in which $a_2^2 - a_1 b_2 = 0$ is the case in which the tangent to the turnpike is vertical.

2) X and Y point to opposite sides of $\Delta_B^{-1}(0)$ that implies $arctan(b_0/a_0) > arctan(m_s)$ where we assume $[0,\pi[$ as range of the function $arctan$.

Let us compute the evolution of the MTF in a neighborhood I of $(t_1'^+, 0)$. We refer to the following figure:



We have to compute the front F_T in the region $\Omega_1 \subset I$ enclosed by γ_{op}^+ (before time $t_1'^+$) and the trajectory γ_0 corresponding to constant control -1 starting from $(t_1'^+, 0)$, on the region Ω_2 enclosed by γ_0 and the turnpike, finally on the region Ω_3 between the turnpike and γ_{op}^+ (after time $t_1'^+$).
In Ω_1 using the expression of $X(x)$ given by formula (13), we obtain for F_T the expression:

$$x_2 = \frac{b_0}{a_0 - 1}(x_1 - T) + O((x_1 - t_1'^+)^2). \tag{14}$$

To compute the locus F_T in Ω_2 and Ω_3 we need the explicit expression for the singular trajectory. The velocity of the singular trajectory at a point $x \in S$ is the convex combination of $X(x)$ and $Y(x)$ corresponding to the direction of the turnpike. This guarantees the smoothness of the MTF on S (see Proposition 5) and gives us the expression for the singular trajectory γ^s ($S = Supp(\gamma^s)$, $\gamma^s(0) = \gamma_{op}^+(t_1'^+)$):

$$\gamma^s(t) = \begin{pmatrix} t_1'^+ + v_1 t + O(t^2) \\ v_2 t + O(t^2) \end{pmatrix} \quad \text{where:}$$

$$v_1 = \frac{b_0\left(a_1\, {b_0}^2 - (-1 + a_0)^2\, b_2\right)}{a_1\, b_0\, \left(-1 + 2\,a_0 - {a_0}^2 + {b_0}^2\right) + (-1 + a_0)^2\, \left((-1 + a_0)\, b_1 - b_0\, b_2\right)} \tag{15}$$

$$v_2 = \frac{(-1 + a_0)\, b_0\, (a_1\, b_0 + b_1 - a_0\, b_1)}{a_1\, b_0\, \left(-1 + 2\,a_0 - {a_0}^2 + {b_0}^2\right) + (-1 + a_0)^2\, \left((-1 + a_0)\, b_1 - b_0\, b_2\right)}. \tag{16}$$

With similar arguments to the ones of the previous subsection we obtain the expression for F_T in Ω_2:

$$x_2 = v_2(T - t_1'^+) + \frac{(x_1 - t_1'^+) - v_1(T - t_1'^+)}{(a_0 - v_1)}(b_0 - v_2) + O((x_1 - t_1'^+)^2).$$

Using (15) and (16), we obtain exactly the expression (14). This means that F_T is $\mathcal{C}^1$ at the intersection point with $Supp(\gamma_0)$. Exactly the same result holds in the case when condition **2)** fails that is when at $(t_1'^+, 0)$ a switching curve bifurcates from γ_{op}^+. In this case, we have to use directly the PMP, but we omit this proof because it is entirely similar to the proof of Proposition 6 below. The smoothness properties of F_T in a neighborhood of $Supp(\gamma_{op}^+|_{]0,t_1'^+]})$ comes immediately from the shape of the synthesis in this region. All these results are summarized in the following:

Proposition 4 *Suppose $t_1'^+ > 0$, fix $y_0 \in Supp(\gamma_{op}^+|_{]0,t_1'^+]})$ and let t_0 be such that $y_0 = \gamma_{op}^+(t_0)$. There exists a ball B with center in y_0 such that if $\varepsilon > 0$ satisfies $\gamma_{op}^+(T) \in B$ for every $T \in]t_0 - \varepsilon, t_0 + \varepsilon[$, then $F_T \cap B$ is a one dimensional piecewise $\mathcal{C}^1$ embedded submanifold of M. Moreover F_T is not $\mathcal{C}^1$ only at the point $x = F_T \cap Supp(\gamma_{op}^+)$ and we have $Int(C^N(\mathcal{R}(T), x)) \neq 0$, that is $\mathcal{R}(T)$ is convex at x*

Analysis up to the First $(YK)_1$ or $(YC)_2$ FP.

Assume $s_1^+ > 0$ (being the case $s_1'^+ > 0$ entirely similar). In this case an abnormal extremal γ_A bifurcates on the right of $\gamma_{op}^+(s_1^+)$ and we can have two kind of FPs:

- $(Y,K)_1$ if an overlap curve starts at $\gamma^+_{op}(s^+_1)$,
- $(Y,C)_2$ if a switching curve starts at $\gamma^+_{op}(s^+_1)$.

The analysis of the first case is postponed to subsection 3.7. Let us consider the second case. Set $x=(x_1,x_2)$, and choose a local system of coordinates in such a way:

$$Y \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad X(x) = \begin{pmatrix} -1 + a_1x_1 + a_2x_2 + O(|x|^2) \\ b_1x_1 + b_2x_2 + O(|x|^2) \end{pmatrix}, \text{where } b_1 < 0, \tag{17}$$

$$X(x) = \begin{pmatrix} c_0 + c_1(x_1 - s^+_1) + c_2x_2 + O(|x-(s^+_1,0)|^2) \\ d_1(x_1 - s^+_1) + d_2x_2 + O(|x-(s^+_1,0)|^2) \end{pmatrix}, \text{ where } c_0 \in]0,1[, \quad d_1 < 0. \tag{18}$$

The expression for X in formula (17) is useful in a neighborhood of $(0,0)$, while the expression (18) gives information in a neighborhood I of $(s^+_1,0)$. The condition that at $(s^+_1,0)$ a C curve starts is:

$$\delta := c_0d_1 - b_1\left(\frac{1-c_0}{2}\right)^2 < 0. \tag{19}$$

Indeed, following [8], one can compute the expression of the curve D reached at the same time by trajectories leaving the origin with different controls. The above condition corresponds precisely to having that extremal trajectories reach the C curve before intersecting D. If the opposite happens D is an overlap curve of the optimal synthesis.

Let $(\bar\gamma,\bar\lambda)$ be an extremal trajectory of (1) corresponding to constant control -1 on the interval $[0,\tau_1[$ and to constant control $+1$ on the interval $[\tau_1,\tau_2]$. We have:

$$\bar\gamma(\tau_1) = \begin{pmatrix} -\tau_1 + O(\tau_1^2) \\ -\frac{1}{2}b_1\tau_1^2 + O(\tau_1^3) \end{pmatrix}, \quad \bar\gamma(\tau_2) = \begin{pmatrix} -\tau_1 + \tau_2 + O(\tau_1^2) \\ -\frac{1}{2}b_1\tau_1^2 + O(\tau_1^3) \end{pmatrix}.$$

Now if τ_1,τ_2 are switching times for $\bar\gamma$ we must have:

$$\bar\lambda(\tau_1)\cdot G(\bar\gamma(\tau_1)) = 0, \tag{20}$$
$$\bar\lambda(\tau_2)\cdot G(\bar\gamma(\tau_2)) = 0. \tag{21}$$

Moreover, from equation **i)** of the PMP, we have $\bar\lambda(\tau_1)=\bar\lambda(\tau_2)=:\bar\lambda$. Set $\bar\lambda=(\bar\lambda_1,\bar\lambda_2)$ and normalize $\bar\lambda$ in such a way that $\bar\lambda_2=\sqrt{1-\bar\lambda_1}$. Equations (20), (21) become:

$$f_1(\lambda_1,\tau_1,\tau_2) := \bar\lambda_1(1+a_1\tau_1) + \sqrt{1-\bar\lambda_1}(\frac{1}{2}b_1\tau_1) + O(\tau_1^2) = 0,$$
$$f_2(\lambda_1,\tau_1,\tau_2) := \bar\lambda_1(\frac{1}{2}(1-c_0) - c_1(-\tau_1+\tau_2-s^+_1)) + \sqrt{1-\bar\lambda_1}(-\frac{1}{2}d_1(-\tau_1+\tau_2-s^+_1)) + O(\tau_1^2) = 0.$$

These are two equations for the variable $(\bar\lambda_1,\tau_1,\tau_2)$ and $(0,0,s^+_1)$ is a solution. The 2×2 Jacobian matrix of partial derivatives of f_1 and f_2 with respect to $(\bar\lambda_1,\tau_2)$ has determinant, at the point $(0,0,s^+_1)$, equal to $-d_1/2$. Under the generic assumption $d_1\neq 0$, we can solve the system in a neighborhood of $(0,0,s^+_1)$ expressing $(\bar\lambda_1,\tau_2)$ as a function of τ_1. This yields:

$$\left.\frac{\partial\tau_2}{\partial\tau_1}\right|_{(0,0,s^+_1)} = -\frac{(b_1(1-c_0)-2d_1)}{2d_1} =: m, \tag{22}$$

hence $\tau_2 = s^+_1 + m\tau_1 + O(\tau_1^2)$ and, under the generic condition $m\neq 1$, the parametric expression for the switching curve starting at $(s^+_1,0)$ is:

$$x_1(\tau_1) = s^+_1 + (m-1)\tau_1 + O(\tau_1^2),$$
$$x_2(\tau_1) = -\frac{1}{2}b_1\tau_1^2 + O(\tau_1^3).$$

If $m \neq 0$, we can use τ_2 as parameter so:

$$x_1(\tau_2) = s_1^+ + \frac{m-1}{m}(\tau_2 - s_1^+) + O((\tau_2 - s_1^+)^2),$$
$$x_2(\tau_2) = -\frac{b_1}{2m^2}(\tau_2 - s_1^+)^2 + O((\tau_2 - s_1^+)^3).$$

Assuming the generic condition $c_0 \neq 1$ and $b_1 \neq 0$, we can express x_2 as a function of x_1:

$$x_2 = -\frac{2d_1^2}{b_1(1-c_0)^2}(x_1 - s_1^+)^2 + O((x_1 - s_1^+)^3). \tag{23}$$

Therefore the switching curve starting at the first switching time is tangent to $Supp(\gamma)$. Now, the expression for the abnormal extremal γ_A can be obtained from (17):

$$\gamma_A(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} s_1^+ + c_0 t + O(t^2) \\ d_1(\frac{1}{2}c_0 t^2) + O(t^3) \end{pmatrix}. \tag{24}$$

Thus we may describe $Supp(\gamma_A)$ as:

$$x_2^{\gamma_A}(x_1) = \frac{1}{2}\frac{d_1}{c_0}(x_1 - s_1^+)^2 + O((x_1 - s_1^+)^3). \tag{25}$$

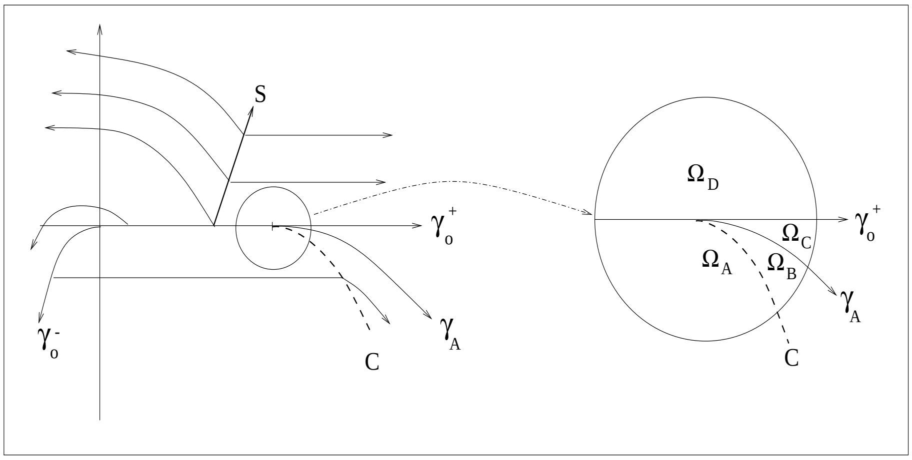


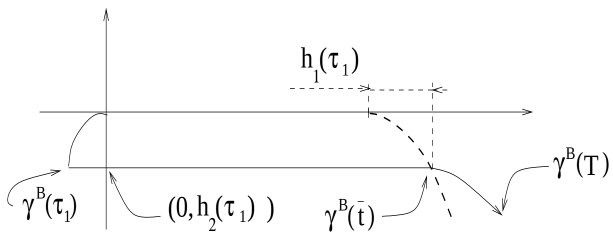
Figure 3.3

Divide I in the four regions Ω_A, Ω_B, Ω_C, Ω_D delimited by γ_{op}^+, C and γ_A as in Figure 3.3. In the following we compute F_T $(T = \tau_1 + \tau_2)$ in these regions and we check that on C and γ_A the tangents to F_T from both sides coincide.

In Ω_A the expression for F_T is obtained similarly to (11):

$$x_2^A(x_1) = \frac{1}{8}b_1(x_1 - T)^2 + O(x_1^3). \tag{26}$$

A point $x^B \in \Omega_B$ is reached by an optimal trajectory γ^B corresponding to constant control -1 in the interval $[0, \tau_1]$, to constant control +1 in the interval $[\tau_1, \bar{t}]$ (where $\bar{t}$ is the time in which this trajectory intersects the C curve) and to constant control -1 in the interval $[\bar{t}, T]$.



Let us parameterize the curve C with τ_1. Using (23) we obtain the expression:

$$\begin{cases} x_1(\tau_1) = s_1^+ + h_1(\tau_1) + O(\tau_1^2) \\ x_2(\tau_1) = h_2(\tau_1) + O(\tau_1^3), \end{cases}$$
$$\text{where } h_1(\tau_1) := \frac{b_1(1-c_0)}{2d_1}\tau_1, \qquad h_2(\tau_1) := \frac{1}{2}b_1\tau_1^2.$$

The expression for an optimal trajectory γ^h of (1) corresponding to constant control -1, and having initial condition $\gamma^h(0) = (s_1^+ + h_1, h_2) \in I$ is:

$$\gamma^h(t) = \begin{pmatrix} x_1(t) = s_1^+ + h_1 + c_0 t + O(t^2, h_1 t, h_2 t) \\ x_2(t) = h_2 + d_1(\frac{1}{2}c_0 t^2 + h_1 t) + O(t^3, h_1 t^2, h_1^2 t, h_2 t^2, h_2^2 t) \end{pmatrix} \tag{27}$$

It follows that $x^B \in F_T \cap \Omega_B$ has coordinates:

$$\begin{cases} x_1^B(\tau_1) = s_1^+ + h_1(\tau_1) + c_0\beta(\tau_1) + O(\tau_1^2) \\ x_2^B(\tau_1) = h_2(\tau_1) + d_1(\frac{1}{2}c_0\beta(\tau_1)^2 + h_1(\tau_1)\beta(\tau_1)) + O(\tau_1^3), \end{cases}$$

where $\beta(\tau_1) := T - \bar{t} = T - h_1(\tau_1) - s_1^+ - 2\tau_1$. Hence the expression for F_T in Ω_B:

$$\begin{aligned} x_2^B(x_1) \;&=\; \frac{1}{2}c_0 d_1(T - s_1^+)^2 + d_1(T - s_1^+)x_1' + d_1\frac{(2-c_0)b_1 + 4d_1}{8\delta}x_1'^2 + O(x_1'^3) \\ &\quad \text{where } x_1' := (x_1 - s_1^+ - c_0(T - s_1^+)). \end{aligned} \tag{28}$$

Finally, the front $F_T \cap \Omega_C$ can be obtained by lengthy but straightforward computations:

$$\begin{aligned} x_2^C(x_1) \;&=\; d_1\frac{x_1'}{1-c_0}(T - s_1^+ - \frac{x_1'}{1-c_0}) + \frac{1}{2}c_0 d_1(T - s_1^+ - \frac{x_1'}{1-c_0})^2 + O((T - s_1^+ - \frac{x_1'}{1-c_0})^3) = \\ &=\; \frac{1}{2}c_0 d_1(T - s_1^+)^2 - d_1(\frac{1}{2}c_0 - 1)(T - s_1^+)\frac{x_1'}{1-c_0} + d_1(\frac{1}{2}c_0 - 1)(\frac{x_1'}{1-c_0})^2 + O((\frac{x_1'}{1-c_0})^3). \end{aligned} \tag{29}$$

Now it is easy to check the following:

- $\dot{x}_2^A(\tilde{x}_1) = \dot{x}_2^B(\tilde{x}_1)$ where $\tilde{x}_1$ is the intersection of the two fronts with the C curve. This means that F_T is $\mathcal{C}^1$ in $\tilde{x}_1$.
- $\dot{x}_2^B(\bar{x}_1) = \dot{x}_2^C(\bar{x}_1) = \dot{x}_2^{\gamma_A}(\bar{x}_1)$ where $\bar{x}_1$ is the intersection of the two fronts with $Supp(\gamma_0)$. This means that F_T is $\mathcal{C}^1$ in $\bar{x}_1$ and it is tangent to γ_A in $\bar{x}_1$.

Analysis of the other Frame Points.
The front F_T at the other singular points along $\gamma_{op}^\pm$ can be analyzed by arguments completely similar to those used above. The only delicate points are $\gamma_{op}^\pm(t_{op}^+)$. At these points we can have a $(Y,K)_{2,3}$ or a $(Y,C)_3$ FP. The first two cases are analyzed in details in the proof of Theorem 1. In the case $(Y,K)_3$ that is the only case in which a γ_k trajectory generates from $\gamma_{op}^\pm$, it is easy to check that the MTF forms a concave corner on γ_k. The last case, can be studied as the interior of strips (see below). We obtain that F_T is always a piecewise $\mathcal{C}^1$ manifold with corners only on $\gamma_{op}^\pm$. Notice that an abnormal extremal may bifurcates from a $(Y,C)_3$ points and in this case we can prove that the front is tangent to the abnormal extremal by arguments similar to those used above.

3.5 The MTF on Other Bases

The front F_T on the singular trajectories is described by the following:

Proposition 5 *Suppose that there exists a point $y^0 \in F_T \cap S$ where S is a turnpike of the optimal synthesis. Then F_T is $\mathcal{C}^1$ at y^0.*

Proof. It is a consequence of the fact that the velocity of the singular trajectory at a point $x \in S$ is the convex combination of $X(x)$ and $Y(x)$ corresponding to the direction of the turnpike. Indeed, fix $y \in F_T$, $y = \gamma_s(T)$ and for every ε small let $x = \gamma_s(T - \varepsilon)$ where γ_s runs the turnpike. Notice that, generically, $\dot{\gamma}_s(T) = \lambda X(y) + (1-\lambda)Y(y)$, for some $0 < \lambda < 1$. Let γ_X^ε be the X–trajectory such that $\gamma_X^\varepsilon(0) = x$ and define γ_Y^ε in the same way with X replaced by Y. We have:

$$\begin{aligned} \gamma_X^\varepsilon(\varepsilon) &= x + X(y)\varepsilon + O(\varepsilon^2) \in F_T \\ \gamma_Y^\varepsilon(\varepsilon) &= x + Y(y)\varepsilon + O(\varepsilon^2) \in F_T. \end{aligned}$$

Now the tangent from the left and from the right to F_T at y are computed using $y - x = \dot{\gamma}_s(T)\varepsilon + O(\varepsilon^2)$: by:

$$\begin{aligned} \frac{d}{d\varepsilon}(\gamma_X^\varepsilon(\varepsilon) - y) &= X(y) - \dot{\gamma}_s(T) = (1-\lambda)(X - Y)(y) \\ \frac{d}{d\varepsilon}(\gamma_y^\varepsilon(\varepsilon) - y) &= Y(y) - \dot{\gamma}_s(T) = -\lambda(X - Y)(y), \end{aligned} \tag{30}$$

proving that F_T is $\mathcal{C}^1$ at y. ■

3.6 The MTF on Open Strips

In this subsection we prove that the MTF restricted to an open strip is always $\mathcal{C}^1$.

Observation 1 Notice that the $\mathcal{C}^1$ property of the MTF can also be obtained reasoning on the position of the covectors with respect to the MTF. More precisely, each covector of the PMP has negative scalar product with all variational vectors corresponding to needle variations. From the structure of the optimal synthesis, it is clear that, fixed an optimal trajectory, the neighboring trajectories of the optimal synthesis can be obtained by needle variations. This implies that the MTF of a strip can not form angles in such a way that the reachable set is locally concave, otherwise this would contradict optimality. On the other side, after the first switching the covector is determined up to the multiplication by a positive constant, hence also angles of the MTF that render the reachable set locally convex are prohibited.

Fix an optimal strip $\mathcal{S}^{a,b,x,f}$ and let $T \in]a, \sup\{f(\alpha), \ \ \alpha \in]a,b[\}[$. We define the strip-MTF (in the following s-MTF) to be the set:

$$F_T^{a,b,x,f} := \{x \in F_T : \ \ x = \gamma_\alpha(T) \text{ for some } \gamma_\alpha \in \mathcal{S}^{a,b,x,f}, \ \ \alpha \in]a,b[\}.$$

Proposition 6 *Let $\mathcal{S}^{a,b,x,f}$ be an optimal strip, $T \in]a, \sup\{f(\alpha), \ \ \alpha \in]a,b[\}[$ and F_T the corresponding s-MTF. Then $F_T^{a,b,x,f}$ is a one–dimensional $\mathcal{C}^1$ embedded submanifold of M. Moreover if $T \in]a, \inf\{f(\alpha), \ \ \alpha \in]a,b[\}[$ then $F_T^{a,b,x,f}$ is also connected.*

The proof follows from the next two Lemmas. With the first we prove that the s-MTF is smooth when it intersects the "first" switching locus. With the second we prove that if the s-MTF is $\mathcal{C}^1$ when it intersect a switching locus, then it is $\mathcal{C}^1$ when it intersects the subsequent. By induction we reach the conclusion.

Lemma 1 *Let $Int(\mathcal{S}^{a,b,x,f})$ be an open optimal strip. Fix $\alpha_0 \in]a,b[$ and suppose that $T_0 \in]\alpha_0, f(\alpha_0)[$ is the first switching time of γ_{α_0} after α_0. Then $F_{T_0}^{a,b,x,f}$ is $\mathcal{C}^1$ in $y^0 := \gamma_{\alpha_0}(T_0)$.*

Proof. Suppose that γ_{α_0} corresponds to constant control $+1$ in the interval $]\alpha_0, T_0[$. Choose a local system of coordinates in such a way that $Y = (1,0)$, $\gamma_{\alpha_0}(\alpha_0) = (0,0)$ and the expression for the base of the strip is:

$$\eta(t) = \begin{pmatrix} 0 \\ t - \alpha_0 \end{pmatrix}, \quad t \in]a,b[.$$

In this system of coordinates we have $y^0 = (T_0 - \alpha_0, 0)$ and the expression of G on the base is ($t \in]a, b[$):

$$G(\eta(t)) = \begin{cases} \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{in the case in which the base is a } X\text{–trajectory, so } X(0,t) = (0,1) \\ \frac{1}{1-\varphi(\eta(t))}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{in the case in which the base is a } Z\text{–trajectory so } (F+\varphi G)(0,t) = (0,1). \end{cases} \tag{31}$$

Let λ_α be the covector associated to γ_α ($\alpha \in]a, b[$). From the condition $\lambda_\alpha(\alpha) \cdot G(\gamma_\alpha(\alpha)) = 0$ we obtain:

$$\lambda_\alpha(\alpha) = \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The switching curve C passing from y^0 is determined by the equation:

$$\lambda_\alpha(t) \cdot G(\gamma_\alpha(t)) = 0. \tag{32}$$

Let t_α be the first switching time of γ_α after α, then $\dot{\lambda}_\alpha(t) = 0$ for $t \in [\alpha, t_\alpha]$. We get that condition (32) is equivalent to:

$$G_1(x) + G_2(x) = 0, \tag{33}$$

where $G_1(x)$ and $G_2(x)$ are the two components of G. In terms of components of the vector field X, (33) can be written as:

$$X_1(x) + X_2(x) = 1. \tag{34}$$

We suppose that at y^0, $F_T^{a,b,x,f}$ and C are not tangent otherwise the conclusion follows easily. Assume that we are in the situation of the following figure (the other case being similar):

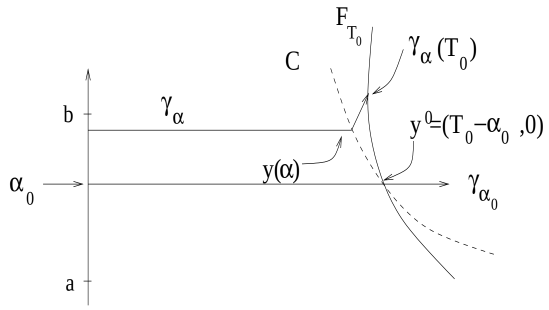


Figure 3.4

In a neighborhood of the point y^0 we may parameterize the C curve by a function:

$$y(\alpha) = \begin{pmatrix} T_0 - \alpha_0 - a(\alpha) \\ \alpha \end{pmatrix}, \tag{35}$$

where $a(.)$ is a $\mathcal{C}^1$ function defined in a neighborhood of $\alpha = 0$ and satisfying $a(0) = 0$. At time T_0, a trajectory with $\alpha > 0$ reaches the point:

$$z(\alpha) := \gamma_\alpha(T_0) = y(\alpha) + (a(\alpha) - \alpha)X(y^0) + O(\alpha^2). \tag{36}$$

It follows

$$\dot{z}(0) = \begin{pmatrix} -\dot{a}(0) + (\dot{a}(0) - 1)X_1(y^0) \\ 1 + (\dot{a}(0) - 1)X_2(y^0) \end{pmatrix}.$$

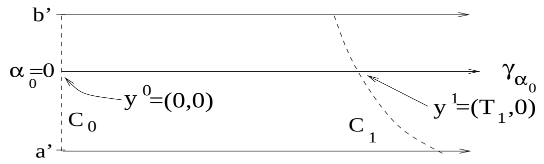
While in the region such that $x_2 < 0$ we have that the s–MTF has direction $(-1, 1)$. The front $F_T^{a,b,x,f}$ is $\mathcal{C}^1$ in y^0 iff $\dot{z}(0)$ is parallel to $(-1, 1)$. This condition is satisfied since (34) holds. ■

Lemma 2 *Let $Int(\mathcal{S}^{a,b,x,f})$ be an open optimal strip. Fix $\alpha_0 \in]a,b[$ and suppose that $T_0, T_1 \in]\alpha_0, f(\alpha_0)[$ are two subsequent switching times of γ_{α_0} after α_0. Suppose that $F^{a,b,x,f}_{T_0}$ is $\mathcal{C}^1$ in $y^0 := \gamma_{\alpha_0}(T_0)$. Then $F^{a,b,x,f}_{T_1}$ is $\mathcal{C}^1$ in $y^1 := \gamma_{\alpha_0}(T_1)$.*

Proof. Suppose that γ_{α_0} corresponds to constant control $+1$ in the interval $]T_0, T_1[$. Set $x = (x_1, x_2)$, choose a local system of coordinates in such a way $Y = (1,0)$, $y_0 = \gamma_{\alpha_0}(T_0) = (0,0)$, the expression for the switching curve C_0 passing from y^0 is:

$$C_0 = \{x \in M : x_1 = 0, \quad x_2 \in]a', b'[\quad a' < 0, \quad b' > 0\},$$

and $X(0, x_2) = (\beta, \beta)$ $(\beta \in \mathbb{R} \setminus \{0\})$ for $x_2 \in]a', b'[$.



In this system of coordinates we have $y^1 = (T_1 - T_0, 0)$ and the expression of G on C_0 is:

$$G((0, x_2)) = \frac{1}{2} \begin{pmatrix} 1-\beta \\ -\beta \end{pmatrix} \text{ for every } x_2 \in]a', b'[. \tag{37}$$

Using the same reasoning as to the proof of the previous Lemma, we have that the switching curve C_1 passing through y^1 is determined by the condition:

$$X_1(x) + \frac{1-\beta}{\beta} X_2(x) = 1. \tag{38}$$

Suppose that we are in the same situation of the proof of Lemma 1 (see figure 3.4). Let $T(\alpha)$ and $\tau(\alpha)$ be the switching times of $\gamma(\alpha)$ on respectively switching curves C_1 and C_2. Define $a(\alpha) = (T_1 - T_0) - T(\alpha) - \tau(\alpha)$ and notice that $a(.)$ is $\mathcal{C}^1$ with $a(0) = 0$. At time T_1 a trajectory with $\alpha > 0$ reaches the point:

$$z(\alpha) := \gamma_\alpha(T_1) = y(\alpha) + (a(\alpha) - \alpha \dot{T}(0)) X(y^1) + O(\alpha^2), \tag{39}$$

where:

$$y(\alpha) = \begin{pmatrix} T_1 - T_0 - a(\alpha) \\ \alpha \end{pmatrix},$$

(being $a(.)$ a $\mathcal{C}^1$ function defined in a neighborhood of $\alpha = 0$ and satisfying $a(0) = 0$) and $T(\alpha)$ is the switching time of γ_α on the switching curve passing from y^0. It follows:

$$\dot{z}(0) = \begin{pmatrix} -\dot{a}(0) + (\dot{a}(0) - \dot{T}(0)) X_1(y^1) \\ 1 + (\dot{a}(0) - \dot{T}(0)) X_2(y^1) \end{pmatrix}.$$

This vector must be parallel to the vector $(-\dot{T}(0), 1)$, i.e. the following conditions must hold:

(c1) $\dot{T}(0) = \frac{1-\beta}{\beta}$ if $\dot{T}(0) \neq 0$,

(c2) $\beta = 1$ if $\dot{T}(0) = 0$.

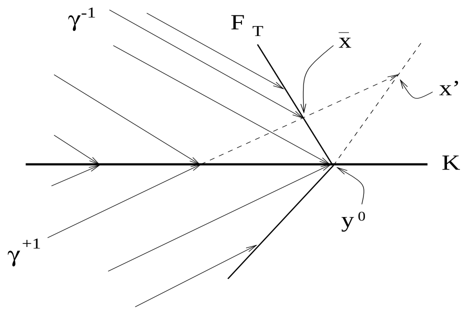
Imposing that $F^{a,b,x,f}_{T_0}$ is $\mathcal{C}^1$ in $y^0 := \gamma_{\alpha_0}(T_0)$, it is easy to check that **(c1)** and **(c2)** are verified. ∎

3.7 F_T on Overlap Curves

The MTF along overlap curves is described by the following:

Proposition 7 *Let $y^0 = F_T \cap K$ where K is some overlap curve of the optimal synthesis and suppose that condition* **(C2)** *(at the end of Section 2) holds. Then there exists a ball B centered at y^0 such that $F_T \cap B$ is not $\mathcal{C}^1$ only at y^0. Moreover, generically, $C^N(\mathcal{R}(T), y^0) = \{0\}$, that is $\mathcal{R}(\mathcal{T})$ is concave at y^0.*

Proof. Choosing B sufficiently small, we have that y^0 is the only possible point in which F_T is not $\mathcal{C}^1$, indeed $(F_T \cap B) \setminus \{y^0\}$ is contained in interiors of strips. By condition **(C2)**, the two connected components of $(F_T \cap B) \setminus \{y^0\}$ cannot be tangent to K. Moreover the K curve is reached from the two sides by trajectories with constant control $+1$ and -1 respectively. By contradiction suppose that we have $Int(C^N(\mathcal{R}(T), y^0)) \neq \emptyset$, that means that we have the situation of the following figure:



The point $\bar{x}$ is reached by a trajectory γ^{-1} (corresponding to constant control -1) at time T, while it is reached at time $T' < T$ by a trajectory γ^{+1} corresponding to constant control +1. This is due to the fact that γ^{+1} reaches x' at time T. This contradicts the optimality of γ^{-1}. The situation in which $(F_T \cap B)$ is $\mathcal{C}^1$ in y^0 is not generic. Thus generically $\mathcal{R}(T)$ is concave at y_0, hence $C^N(\mathcal{R}(T), y^0) = \{0\}$. ∎

3.8 F_T on Borders

In this section we study the smoothness property of the MTF on strip borders. In the first and second subsections we analyze respectively the strip borders of kind γ_0 and γ_A. In the third subsection we analyze the strip borders of kind γ_k. Since at the two sides of a border of a strip the trajectories have the same control, we have to analyze only the FPs at which this border generates. The FPs of kind $(Y, .)$ are already studied in subsection 3.4. Here we complete the analysis.

3.8.1 F_T on Strip Border of Kind γ_0

The strip borders of kind γ_0, that do not bifurcate from $\gamma_{op}^{\pm}$, are generated at the following FPs: $(C, C)_1$, $(C, S)_{1,2}$, $(S, K)_1$, and at $(C, C)_2$ if $\Delta_A \neq 0$ (see Figure 2.1). We analyze in details the case $(C, S)_1$ and $(S, K)_1$. The other cases can be treated by the same argument used in subsection 3.5

Assume to have a $(C, S)_1$ or a $(S, K)_1$ FP at $y^1 := \gamma_0(t_1)$ and suppose the following:

- the previous switching time of γ_0 is $t_0 < t_1$;
- γ_0 corresponds to constant control $+1$ in the interval $[t_0, t_1]$;
- $y^0 := \gamma_0(t_0)$ belongs to a X FC (being similar the cases in which it belong to a switching curve or to a turnpike).

Set $T_0 = t_1 - t_0$. Choose a system of coordinates and shift the time in such a way that $y^0 = \gamma_0(0) = 0$, and:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(0, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X(x) = \begin{pmatrix} a_0 + a_1(x_1 - T_0) + a_2 x_2 + O((x_1 - T_0)^2, x_2^2) \\ b_0 + b_1(x_1 - T_0) + b_2 x_2 + O((x_1 - T_0)^2, x_2^2) \end{pmatrix}, \quad a_0 < 1, \quad b_0 > 0.$$

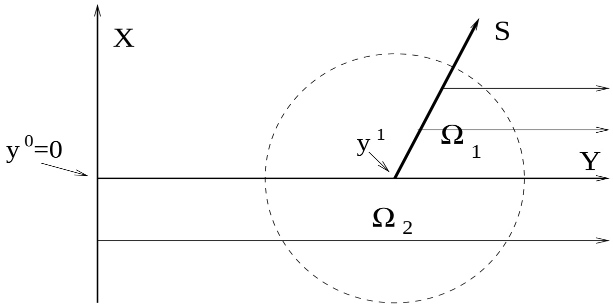


Figure 3.5

To have a $(C,S)_1$ or a $(S,K)_1$ FP at y^1 the following condition must hold (cfr. subsection 3.4):

$$a_2 = \frac{a_1 b_0}{a_0 - 1}. \tag{40}$$

Exactly as in subsection (3.4), we obtain the equation of the singular trajectory starting from y_1 at time 0:

$$\gamma^s(t) = \begin{pmatrix} T_0 + v_1 t + O(t^2) \\ v_2 t + O(t^2) \end{pmatrix},$$

where v_1 and v_2 are given by formulas (15) and (16). Let I be a neighborhood of y^1 and let us compute the MTF in the region Ω_1 enclosed by γ_0 (after T_0) and S (see figure 3.5). Consider a trajectory corresponding to constant control $+1$ in $[0, T_0[$, to the singular control $\varphi(\gamma^s(t - T_0))$ on $[T_0, T_0 + \alpha[$ and to constant control $+1$ on $[T_0 + \alpha, T]$. At time T this trajectory reaches the point:

$$z(\alpha) = (T_0, 0) + \alpha \dot{\gamma}^s(0) + (T - T_0 - \alpha) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(\alpha^2) \in \Omega_1.$$

It follows:

$$\dot{z}(0) = \dot{\gamma}^s(0) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The MTF is $\mathcal{C}^1$ at $\gamma_0(T)$ iff $\dot{z}(0)$ is parallel to $(1, -1)$, so we have to check the condition:

$$v^1 + v^2 = 1. \tag{41}$$

Let $\lambda(.)$ be the covector associated to $\gamma_0(.)$. From the following equations:

$$\begin{aligned} &\lambda(0) \cdot G(\gamma_0(0)) = 0, \\ &\dot{\lambda}(t) = 0 \quad \forall\, t \in [0, T_0], \\ &\lambda(T_0) \cdot G(\gamma_0(T_0)) = 0, \\ &\lambda(T_0) \cdot [F, G](\gamma_0(T_0)) = 0, \end{aligned}$$

we obtain:

$$a_0 + b_0 = 1, \qquad a_1 + b_1 = 0. \tag{42}$$

Using (42) and (40) one checks immediately that condition (41) is verified.

3.8.2 F_T on Strip Border of Kind γ_A and Proof of Theorem 4

After the first switching, the only FP involving an abnormal extremal is the point $(C,C)_2$ where Δ_A vanishes. In the following we prove that if t_n, t_{n+1} are two consequent switching times for an abnormal extremal γ_A, and the switching curve passing through t_n is tangent to γ_A then the switching curve passing through t_{n+1} is tangent to γ_A as well. From this fact, with an analysis completely similar to that of subsection 3.6, it follows that the MTF is $\mathcal{C}^1$ on γ_A and it is always tangent to $Supp(\gamma_A)$. This proves Theorem 4.

Let us consider only the case in which γ_A corresponds to constant control $+1$ on $[t_n, t_{n+1}]$, the opposite case being similar. Choose a local system of coordinates and rescale the time in such a way that:

$$\begin{aligned}
&t_n = 0, \quad \gamma_A(t_n) = 0,\\
&Y \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad X(x) = \begin{pmatrix} e_0 + e_1 x_1 + e_2 x_2 + O(|x|^2) \\ f_1 x_1 + f_2 x_2 + O(|x|^2) \end{pmatrix},\\
&X(x) = \begin{pmatrix} g_0 + g_1(x_1 - t_{n+1}) + g_2 x_2 + O(|x - (t_{n+1}, 0)|^2) \\ h_1(x_1 - t_{n+1}) + h_2 x_2 + O(|x - (t_{n+1}, 0)|^2) \end{pmatrix}.
\end{aligned}$$

We can parameterize the switching curve C_n through $\gamma_A(t_n)$ in the following way. Given $x \in C_n$ there exists (γ_x, λ_x) extremal that switches at x at time $\tau_n(x)$. By induction, we can assume that τ_n is invertible for x near $\gamma(t_n)$ and parameterize C_n by τ_n:

$$\begin{aligned}
x_1(\tau_n) &= \alpha_n \tau_n + O(\tau_n^2),\\
x_2(\tau_n) &= \beta_n \tau_n^2 + O(\tau_n^3),
\end{aligned}$$

for some α_n, β_n that by genericity we may assume different from zero.
Let $(\tilde{\gamma}, \tilde{\lambda}) := (\tilde{\gamma}, \tilde{\lambda})_{\tau_n}$ be the extremal trajectory of (1) switching at τ_n on C_n and let τ_{n+1} be the next switching time. We have:

$$\tilde{\gamma}(\tau_n) = \begin{pmatrix} \alpha_n \tau_n + O(\tau_n^2) \\ \beta_n \tau_n^2 + O(\tau_n^3) \end{pmatrix}, \quad \tilde{\gamma}(\tau_{n+1}) = \begin{pmatrix} \alpha_n \tau_n + \tau_{n+1} + O(\tau_n^2) \\ \beta_n \tau_n^2 + O(\tau_n^3) \end{pmatrix}.$$

and:

$$\tilde{\lambda}(\tau_n) \cdot G(\tilde{\gamma}(\tau_n)) = 0, \tag{43}$$

$$\tilde{\lambda}(\tau_{n+1}) \cdot G(\tilde{\gamma}(\tau_{n+1})) = 0. \tag{44}$$

With computations entirely similar to the those of subsection 3.8.1 and using similar generic conditions we conclude that the switching curve passing through $\gamma(t_{n+1})$ has the expression:

$$\begin{aligned}
x_1(\tau_{n+1}) &= t_{n+1} + \alpha_{n+1}(\tau_{n+1} - t_{n+1}) + O((\tau_{n+1} - t_{n+1})^2),\\
x_2(\tau_{n+1}) &= \beta_{n+1}(\tau_{n+1} - t_{n+1})^2 + O((\tau_{n+1} - t_{n+1})^3),
\end{aligned}$$

for some $\alpha_{n+1}, \beta_{n+1} \neq 0$. This concludes the proof.

3.8.3 F_T on Strip Border of Kind γ_k

The strip borders of kind γ_k, that do not bifurcate from $\gamma_{op}^{\pm}$, are generated at the following FPs: $(C,K)_1, (S,K)_2$. Again it is easy to check that the MTF forms a corner on γ_k. Moreover, for every $x \in \gamma_k$, one has $C^N(\mathcal{R}(\mathbf{T}(x)), x) = \{0\}$ i.e. the reachable set is locally concave at x. For example in the $(C,K)_1$ case, using again the PMP, one checks that the front is not deflected when passing through the C curve. Since the front forms a concave angle on K, the same happens on γ_k.

3.9 Proof of Theorem 3

The front at points in the interior of a strip or a strip border has been analyzed in the previous sections.

Assume that x is a Frame Point and **(C1)** holds, so x is not of kind $(Y,K)_{2,3}$ or (K,K). The cases (Y,X), $(Y,C)_{1,2,3}$, (Y,S) and $(Y,K)_1$ have been treated in subsection 3.3. The cases $(C,C)_1$, $(C,S)_{1,2}$, $(S,K)_1$ and $(C,C)_2$ with $\Delta_A \neq 0$, were analyzed in subsection 3.8.1. The case $(C,C)_2$ with $\Delta_A = 0$ is in subsection 3.8.2. Finally, the cases $(C,K)_1$ and $(S,K)_2$ are in subsection 3.8.3. The remaining case $(C,K)_2$ is easily treated by using Proposition 6 and 7.

Finally, if x is on a frame curve of kind K, that is where a strip may end, and the condition **(C2)** holds then we can apply Proposition 7.

This concludes the proof of Theorem 3. ■

3.10 The Extremal Front

If we consider all extremal trajectories, including the not optimal ones, we can define the extremal front F_T^e as the set of points that are reached by an extremal trajectory at time T. Obviously we have $F_T \subset F_T^e$. Our aim is to study the singularities developed by F^e as a one dimensional wave front.

It is well known that for one dimensional wave fronts there is only a generic kind of singularity called *standard singularity,* see [2, 3]. We want prove that at Frame Points of the optimal synthesis the wave front F^e develops only standard singularities.

We describe the situation only at the Frame Point $y^0 = \gamma_{op}^+(s_1^+)$ that is of kind $(Y,K)_1$ being the other cases $(C,K)_2$ and $(S,K)_1$ similar. The extremal synthesis, that is the collection of all extremals in the cotangent bundle (see [6] for details), at $(Y,K)_1$ has a cusp projection singularity. The borders of the cusp are the X trajectory from y^0 (that is a γ_0 Frame Curve for the extremal synthesis) and a curve of kind $\bar{C}$, that is a curve of switching points reached by not optimal extremal trajectories that reflect "back" after the switching.

Let $x = (x_1, x_2)$ and choose a system of coordinates in such a way (17) and (18) hold (but with the condition $c_0 \in]0,1[$ replaced by $c_0 < 1$). The fact that at $(s_1^+, 0)$ a K curve starts implies (see the analysis of the point $(Y,C)_2$ and formula (19)):

$$\delta := c_0 d_1 - b_1 \left(\frac{1 - c_0}{2} \right)^2 > 0. \tag{45}$$

For time $t < \mathbf{T}(y^0)$, near $\mathbf{T}(y^0)$, $F_t^e = F_t$ and the front F_t^e is a one dimensional piecewise smooth manifold. For time $t > \mathbf{T}(y^0)$, near $\mathbf{T}(y^0)$, the front F_t^e is formed by three branches:

- the first corresponds to trajectories that bifurcate from γ^+ after x_0. Its expression as a function of x_1 is an arc of the function $x_2^A(x_1)$ defined in (26);
- the second corresponds to trajectories that switched on the $\bar{C}$ curve. Its expression as a function of x_1 is an arc of the function $x_2^B(x_1)$ defined in (28);
- the third corresponds to Y trajectories that did not yet reach the curve $\bar{C}$. Its expression as a function of x_1 is an arc of the function $x_2^C(x_1)$ defined in (29).

Let α^A, α^B, α^C be the coefficient of x_1^2 in the expression of x_2^A, x_2^B, x_2^C respectively. With our choice of parameters (i.e. $b_1,\ d_1 < 0,\ c_0 < 1,\ \delta > 0$), one can check that $\alpha^A < 0, \quad \alpha^B > \alpha^C > 0$. Moreover $x^A(x_1)$, $x^B(x_1)$ have a first order tangency at the point:

$$x^{AB} = \frac{(c_0 - 1)b_1 t + 4 d_1 s_1^+}{(c_0 - 1)b_1 + 4 d_1},$$

while $x^B(x_1)$ and $x^C(x_1)$ have a first order tangency between them and with the trajectory exiting y^0 with constant control -1 at the point

$$x^{BC} = s_1^+ + c_0(t - s_1^+).$$

We have $x^{AB} > x^{BC}$ thus the front F^e develops a standard singularity.

Notice that all standard singularities of F^e are developed at Frame Points (namely $(Y,K)_1$, $(C,K)_2$, and $(S,K)_1$) corresponding to cusp projection singularities of the extremal synthesis (see [6]). Figure 3.6 shows the extremal synthesis and F_t^e near the point $(Y,K)_1$, in the two cases in which $c_0 > 0$ and $c_0 < 0$. The case $c_0 = 0$ is not generic.

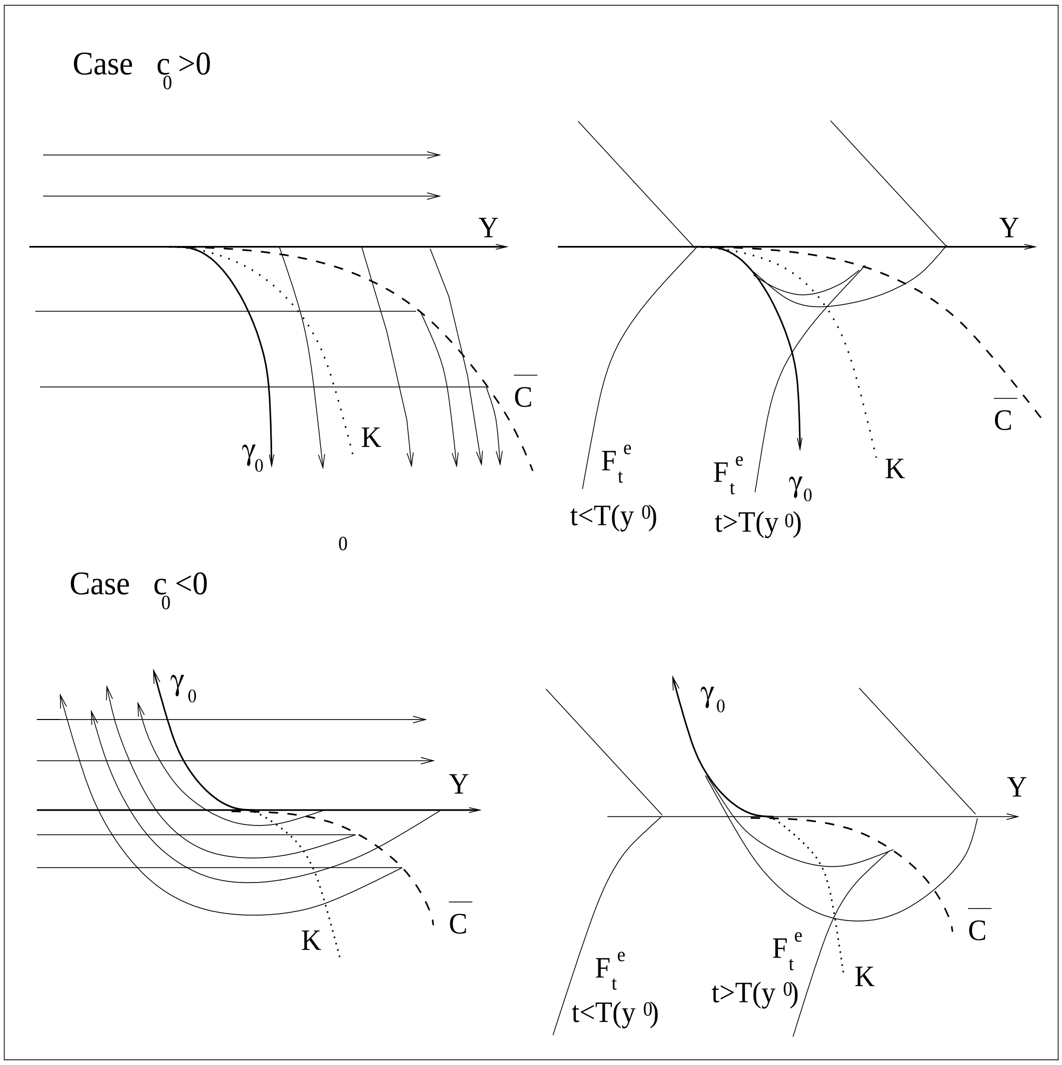


Figure 3.6: The extremal synthesis and the MTF in a neighborhood of the first FP of kind $(Y, K)_1$

4 Proof of Theorem 1

We are going to prove Theorem 1 analyzing the MTF first at points inside a strip, then at points on Frame Curves and finally at Frame Points.

If x is a point inside a strip, not on a C curve, the wave fronts are locally smooth one dimensional manifolds. Moreover, the optimal synthesis is locally a *flow box* of one of the vector fields X and Y. It easily follows that the minimum time function $\mathbf{T}$ is smooth and x is not a critical point (see Figure 1.1), since $\mathbf{T}$ increases in the direction of X or Y.

Assume now that x belongs to a Frame Curve (but is not a Frame Point). If the Frame Curve is of kind X, Y, γ_0 or S then the front is a piecewise $\mathcal{C}^1$ manifold and the minimum time is increasing along the Frame Curve, hence the function $\mathbf{T}$ is locally topologically equivalent to a nonzero linear function and x is not a critical point. Notice that indeed the front is $\mathcal{C}^1$ in case of γ_0 and S curves. If the Frame Curve is of kind C then the front is $\mathcal{C}^1$ and there is an increasing direction (indeed two directions X and Y) so $\mathbf{T}$ is $\mathcal{C}^1$ and topologically equivalent to a nonzero linear function and x is not a critical point. We are left with the cases of K and γ_k Frame Curves. Given a Frame Curve D of K type, there exist two Frame Curves D_1 and D_2 such that the curve D is determined by the solution of the following system, see [8]:

$$\begin{cases} (exp\ tX)\,\alpha_1(s) = (exp\ t'Y)\,\alpha_2(s') \\ s+t = s'+t' \end{cases},$$

where $(exp\ tV)\bar{x}$ indicates the integral trajectory at time t of the vector field V starting from $\bar{x}$ and α_1 and α_2 are two smooth parameterizations of the curves D_1 and D_2 respectively. It is clear that under generic assumptions on X and Y, hence on F and G, the above system determines a smooth curve D such that the minimum time function $\mathbf{T}$ restricted to D is a Morse function. We have now three cases:

a) x is not critical for $\mathbf{T}$ on D,

b) x is a maximum for $\mathbf{T}$ on D,

c) x is a minimum for $\mathbf{T}$ on D.

In case a), we are in the situation of Proposition 7, the MTFs near x are piecewise $\mathcal{C}^1$ manifolds. The function $\mathbf{T}$ is increasing (or decreasing) in the direction tangent to K and is topologically equivalent to a nonzero linear function. Therefore x is not a critical point.

Assume case b) holds, then x is not a Frame Point and condition **(C2)** is violated. Since $\mathbf{T}$ is increasing along the two strips ending at K, x is a maximum for $\mathbf{T}$. The MTF through x reduces to a point and is homeomorphic to a circle for points in a neighborhood of x. Therefore $\mathbf{T}$ is topologically equivalent to $f(x_1,x_2) = -x_1^2 - x_2^2$ at the origin, that is x is topologically a maximum.

If c) holds, then either x is a Frame Point or x is not a Frame Point and condition **(C2)** does not hold. Assume the latter occurs. Again, $\mathbf{T}$ is increasing along the two strips ending at K. The MTF at x is formed by two tangent parabolas through x, for points in a neighborhood is formed by two branches that are smooth if they do not intersect the K curve. Therefore $\mathbf{T}$ is topologically equivalent to $f(x_1,x_2) = x_1^2 - x_2^2$ at the origin, that is x is topologically a regular saddle.

If the Frame Curve is of kind γ_k then we are in case a) above, indeed $\mathbf{T}$ is increasing along γ_k, and we conclude in the same way.

We are left with the case of x Frame Point. The points along $\gamma^{\pm}$ have been analyzed above and the minimum time function $\mathbf{T}$ is topologically equivalent to a linear function at those points. The same happens for the Frame Points of kind $(C,C)_{1,2}$, $(C,S)_{1,2}$, $(C,K)_{1,2}$, $(S,K)_{1,2}$. It remains to analyze the cases $(Y,K)_{2,3}$ and (K,K).

Assume x is a $(Y,K)_2$ Frame Point $x = \gamma^+(\mathbf{T}(x))$ (being the case of γ^- entirely similar). For $\varepsilon > 0$ sufficiently small, there are two strips $\mathcal{S}^{1,2}$ containing $\gamma^+|[\mathbf{T}(x)-\varepsilon, \mathbf{T}(x)]$ and one strip $\mathcal{S}^3$ reaching x from the other side. We indicate by $\mathcal{R}_+$ the reachable set corresponding to trajectories of $\mathcal{S}^{1,2}$ and $\mathcal{R}_*$ that corresponding to trajectories of $\mathcal{S}^3$. Generically, we have that $Int(C^N(\mathcal{R}_+(\mathbf{T}(x)),x)) \neq \emptyset$ and $C^N(\mathcal{R}_*(\mathbf{T}(x)),x)$ is a half line. Hence x is the first point reached on the overlap curve K and is a minimum for $\mathbf{T}$ restricted to K. The front $F_{\mathbf{T}(x)}$ is formed by a smooth branch on the strip $\mathcal{S}^3$ and by a piecewise $\mathcal{C}^1$ branch with an angle at x on the strips $\mathcal{S}^{1,2}$. For times $t < \mathbf{T}(x)$, near $\mathbf{T}(x)$, the front F_t it consists of one smooth branch on $\mathcal{S}^3$ and one piecewise smooth on $\mathcal{S}^{1,2}$, while for $t > \mathbf{T}(x)$, near $\mathbf{T}(x)$, consists of two piecewise smooth branches crossing K on opposite sides of x. We conclude that x is topologically a regular saddle for $\mathbf{T}$.

The analysis of the point $(Y,K)_3$ can be done in a similar way. Notice that x is an endpoint of K and one easily checks that x is a minimum for $\mathbf{T}$ restricted to K. However, now there are four strips intersecting at x. Two of them ends at K coming from different directions, while the other two have the γ_k curve through x as border with γ^+ being the base of one of the two. Again, we conclude that x is topologically a regular saddle for $\mathbf{T}$.

Assume x is of kind (K,K). If x is reached by a γ_0 Frame Curve then the fronts on each side of K are $\mathcal{C}^1$ and generically we are in case a) above, where D now is the union of the two K curves, and $\mathbf{T}$ is equivalent to a nonzero linear function.

If x is reached by a γ_k Frame Curve, then the corresponding front has an angle on γ_k and the reachable set is locally concave, see Section 3.8.3. The function $\mathbf{T}$ has a maximum at x and x is topologically a maximum.

Thus, on $\mathcal{R}(\mathcal{T})$, $\mathbf{T}$ is topologically a Morse function with the set Q being the collection of all Frame Curves of $Int(\mathcal{R}(\mathcal{T}))$.

5 Semiconcavity for a two dimensional target on the plane

The aim of this section is to prove semiconcavity of the minimum time function for planar systems with a two dimensional manifold with boundary, instead of a point, as starting set.

Definition 11 *A function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^2$ open, is said to be semiconcave if for every compact convex $\mathcal{K} \subset A$, there exists $c_{\mathcal{K}} > 0$ such that $f(x) - c_{\mathcal{K}} |x|^2$ is concave on $\mathcal{K}$.*

If we fix a two dimensional manifold N with boundary as starting set then, from the transversality condition of Pontryagin Maximum Principle, the covectors of extremal trajectories at initial time must belong to the normal space to ∂N. Moreover we assume the following local controllability property:

(LC) For every $x \in \partial N$ there exists a control $u \in [-1, 1]$ such that $F(x) + uG(x)$ points inside N.

Theorem 7 *Let N be a smooth two dimensional manifold with boundary and assume that* **(LC)** *holds. Then the minimum time function $\mathbf{T}_N$, with initial set N, is semiconcave.*

Proof. The assumption **(LC)** ensures that the minimum time function $\mathbf{T}_N$ is Lipschitz continuous. The structure of extremal trajectories and singularities of extremal synthesis have been described in [10]. The main difference for the optimal synthesis is that now there are no more the $\gamma^{\pm}$ trajectories and the relative Frame Points. In an entirely similar manner as above, one can prove that generically the minimum time function is topologically a Morse function and the Minimum Time Front has the same properties as before.

Obviously a smooth function is semiconcave, hence we have only to consider Frame Curves and Frame Points. It is easy to check that points, at which $\mathbf{T}_N$ is topologically equivalent to a linear function, satisfy the assumptions of semiconcavity. The same happens for maxima.

Thus we have only to check minima and saddles. The only minimum of the starting point case is the starting point itself and now it is replaced by ∂N; there is no other minimum. Moreover, from the previous analysis we know that the possible saddle points are internal points of overlap curves K or Frame Points of kind $(Y, K)_{2,3}$. In the first case the function $\mathbf{T}_N$ is smooth along K and again we retrieve semiconcavity, while the second case does not happen (there are no $\gamma^{\pm}$ curves now). ■

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