Extremal Synthesis and Morse Property for Minimum Time

CANDIDATE

Ugo Boscain

SUPERVISOR

Prof. Benedetto Piccoli

Thesis submitted for the degree of
"Doctor Philosophiae"

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Contents

Introduction

In this Thesis we study a class of bidimensional optimal control problems that is sufficiently general to be used for some applications but simple enough to reaches a deep understanding of the geometry underlying the Pontryagin Maximum Principle (in the following PMP). We deal with the minimum time stabilization problem to the origin for the planar single-input control system:

$$\dot{x} = F(x) + uG(x), \qquad x \in \mathbb{R}^2, \qquad u \in [-1, 1], \qquad F(0) = 0, \tag{1}$$

where F and G are two $\mathcal{C}^\infty$ vector fields on the plane. For us, a solution to the problem is given by an optimal synthesis, that is a way of associating a unique optimal trajectory to every initial point of the plane. Under generic conditions, the optimal synthesis happens to be generated by a piecewise smooth feedback related to a stratification of $\mathbb{R}^2$, see [12, 19, 38]. This problem was also studied in 70's by Baitmann who analyzed the controllability and the position of switching curves (see [5, 6]).

The concept of regular synthesis was introduced in the pioneering work of Boltianski [7], see also [20] and [35]. For a discussion of the concept of synthesis and its relation to piecewise smooth feedbacks we refer the reader to [33].

The well known Pontryagin Maximum Principle (see [25, 34, 39]) gives a first order condition for the optimal trajectories: every time optimal trajectory has a lift to the cotangent space, formed by vector-covector pairs, satisfying a maximization condition of a suitable Hamiltonian (see Chapter 1 for details). The properties of extremal trajectories, that is trajectories satisfying the PMP, have been studied in details in [36, 37]. Moreover, in a series of papers [17, 19, 31, 32], the existence of a structurally stable optimal synthesis for generic smooth systems was proved and complete classifications of the corresponding (non smooth) flows and relative singularities were given, in the same spirit of the work of Peixoto for two dimensional dynamical systems (see [30]).

In all papers, the authors use the PMP to narrow the class of possibly optimal trajectories. One proves that the set of extremal trajectories can be described by a finite dimensional reduction, more precisely every extremal trajectory is a finite concatenation of arcs that are either bang (corresponding to a constant control equal to ± 1) or singular (corresponding to a smooth feedback defined on some special curves), with a bound on the number of arcs for a fixed time $\tau > 0$. Then, the construction of the (essentially unique) optimal synthesis consists in considering all extremal trajectories with a fixed number of arcs and operating some "cuts" to exclude the not optimal ones.

In Chapter 2, we propose a new approach. This amounts to construct all extremals, that is trajectory-covector pairs satisfying the PMP, and projecting the obtained set onto the plane. This second method is in this case more involved, but induces a more clear understanding of the properties of extremal trajectories and the relation with the minimum time function. In particular the "cuts", called overlaps in [32], appearing in the optimal synthesis are related to projection singularities of the set of extremals, being locally optimal the extremal trajectories coming from both sides of the overlap. In this aspect, our work is related to the recent interesting analysis of projection singularities of smooth families of extremal pairs, proposed in [26].

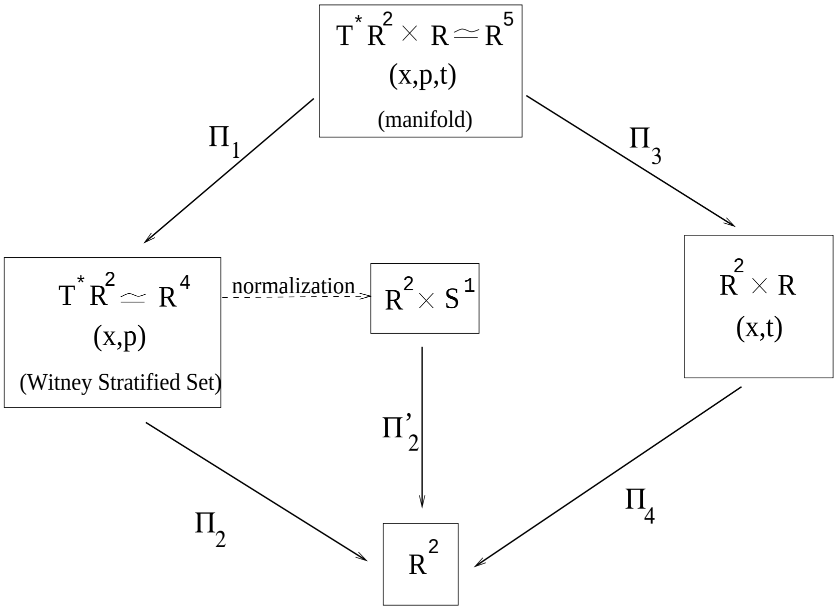
We start constructing all the extremal pairs via an algorithm that terminates, under generic conditions on the pair (F, G), in a finite number of steps. We consider the equivalent problem of starting from the origin and reaching all points of the plane in minimum time. Generically all extremal trajectories leave the origin with control $+1$ or -1. Thus, if $\gamma^{\pm}$ are the extremal arcs corresponding to constant controls ± 1 and exiting the origin, then each extremal trajectory bifurcates from $\gamma^{\pm}$. The set of bifurcating extremal trajectories are divided into "strips", that are one dimensional families presenting the same sequence of arcs (bang and singular). Then the evolution of "strips" is studied under generic conditions. The obtained set of extremal pairs has some regularity properties summarized in the definition of extremal synthesis, see Section 2.1. The set $\mathcal{N}$ of pairs that are reached by an extremal trajectory fails to be a manifold but is still a Withney stratified subset of $T^*\mathbb{R}^2 = \mathbb{R}^4$ of dimension 3. Moreover, the extremal trajectories (in finite time) are again finite concatenations of bang and singular arcs and the set formed by points at which $\mathcal{N}$ fails to be a manifold is the union of a countable collection of two dimensional strata. The covector associated to an extremal pair is defined up to a positive multiplication constant, hence we project $\mathcal{N}$ onto $\mathbb{R}^2 \times S^1$ obtaining a stratified set N of dimension two:

$$N = \{(x, p) \in \mathbb{R}^2 \times S^1 : \text{there exists a time } t > 0 \text{ and an extremal pair } (\gamma, \lambda) : [0, t] \to \mathbb{R}^2 \times S^1 \text{ such that } \gamma(0) = 0,\ \gamma(t) = x,\ \lambda(t) = p\}.$$

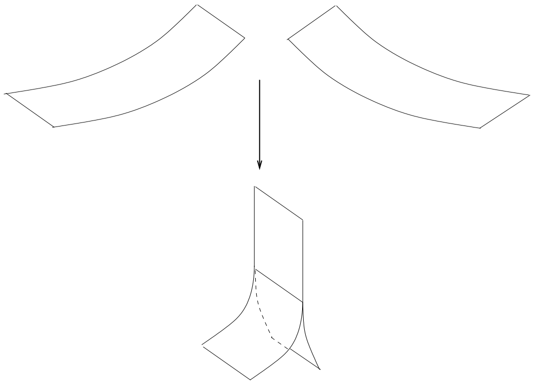
Then we study the stable singularities of the projection of N onto $\mathbb{R}^2$. At points where N is a manifold, we are in the situation of the projection of a two dimensional manifold on the plane, hence we may expect to find the classical Whitney singularities of smooth maps between two dimensional manifolds, namely folds and cusps. Apart from some special singularities (vertical, see Theorem 3, Chapter 2) due to the fact that the target is a point, the stable projection singularities encompass, beside folds and cusps, two new singularities called bifold and ribbon. The first appears in relation with not optimal trajectories along fold points, while the second appears only on abnormal extremals. The latters are extremal trajectories with vanishing Hamiltonian, a detailed analysis of which is in Chapter 3. More structure of N near the origin is studied in Chapter 4. It is necessary to point out that being N only a stratified set and not a smooth manifold, the classification of stable projection singularities is to be intended in topological sense.

The strata of dimension one and zero of N are called Frame Curves and Points. As a byproduct we get a complete classification of Frame Curves and Points, see Section 2.6.

Let us give some more details for a better understanding of the situation. Consider the space of the three variable (x, p, t) where $(x, p) \in T^*\mathbb{R}^2 = \mathbb{R}^4$ and t is the time. We refer to the following picture:

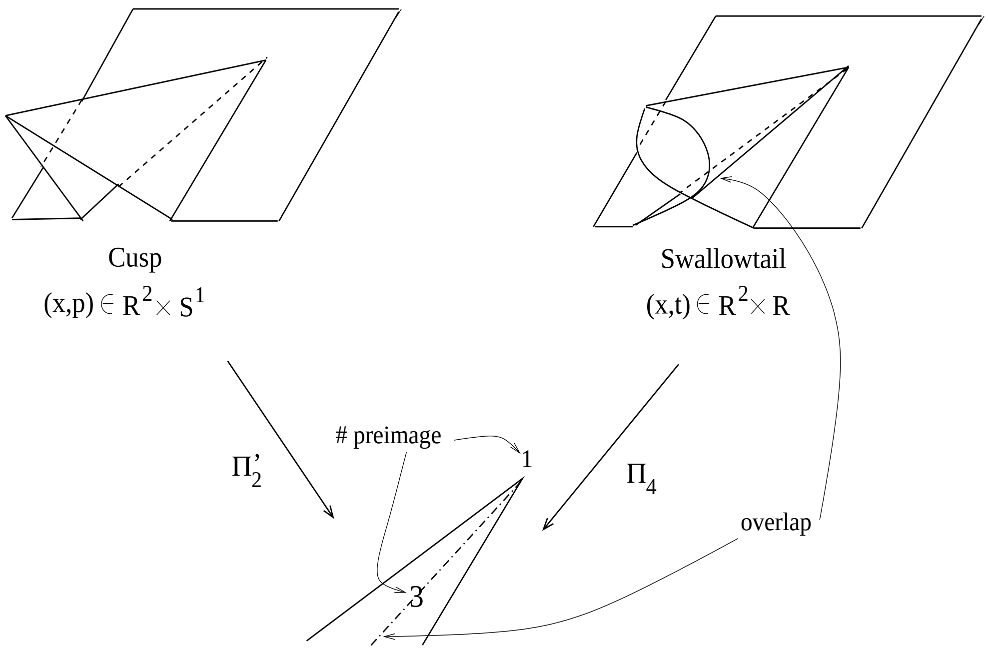


The PMP determines a set in the (x, p, t) space that is a three-dimensional piecewise $\mathcal{C}^1$ manifold. Projecting in the (x, p) space through Π_1, it may happen that two regions glue together:



This is the reason why N is a Whitney stratified set, but not a manifold. The situation is described by Theorem 9 Chapter 2. The singularities of the projection Π_2' are those classified in Chapter 2 (cusp, fold, vertical, bifold, ribbon).

On the other side also the projection Π_3 on the (x, t) space and Π_4 on the x space are interesting. Every cusp in $\mathbb{R}^2 \times S^1$ corresponds to a swallowtail in the (x, t) space and this is due to the presence of an overlap curve in the optimal synthesis:



In Chapter 3 we analyze the generic properties of abnormal extremals. We prove that these are finite concatenations of bang arcs (that is corresponding to constant ± 1 control). Moreover, the switchings (discontinuity points of the control) happens exactly when the abnormal extremal crosses the set of zeroes of the function $\Delta_A = F \wedge G$ that is when the two vector fields F and G are collinear. In many cases, an abnormal extremal is formed by fold points, that is there are two sheets of the cotangent bundle covered by extremal trajectories that projects onto the same region of the plane and the boundary of this region is the projection of the abnormal extremal. All possible generic singularities of the synthesis on the plane occurring along (projections of) abnormal extremals are classified in Chapter 2. However, one has to prove that all singularities indeed appears for some generic system, in particular those corresponding to the new projection singularities bifold and ribbon. The set of possible singularities is formed of 28 (equivalence classes of) singular points, but not all sequences of singularities can be realized.

We recall that a set of words Ω from a given finite alphabet is recognizable if there exists an automaton that generates exactly the words of Ω. An automaton is roughly speaking a graph, with labelled edges, and one constructs words considering all the paths starting from a given set of points and ending to another fixed set of points. An important role in theoretical computer science is played by recognizable sets of words (see [24]). We are able to prove that the generic sequences of singularities along abnormal extremals can be classified with a recognizable set of words. As a by product, we obtain the existence of systems presenting singular points corresponding to projection singularities of bifold and ribbon type.

In Chapter 4 we are concerned with the *initial conditions set* N_0:

$$N_0 = \{(x,p) \in \mathbb{R}^2 \times S^1 : \text{there exists a time } t>0 \text{ and an extremal pair } (\gamma,\lambda) : [0,t] \to \mathbb{R}^2 \times S^1 \text{ such that } \gamma(0)=0,\ \gamma(t)=x,\ \lambda(0)=p\}.$$

We determine explicitly this locus in a neighborhood of the origin and we give a sufficient conditions under which it is a piecewise–$\mathcal{C}^1$ manifold with boundary. The set N_0 is more easily computed than N since, outside $Supp(\gamma^+) \cup Supp(\gamma^-)$, the extremal trajectories are level surfaces of N_0. Moreover, N_0 fails to be a manifold, but its singularities are quite different from those of N. If $I(0)$ denotes a neighborhood of the origin in $\mathbb{R}^2 \times S^1$ then the knowledge of $N_0|_{I(0)}$ is permits to determine the properties of $N|_{I(0)}$.

Notice that the analysis of Chapter 2, 3 and 4 relies on local arguments, hence the results hold in the more general case of a smooth two dimensional manifold M.

As an application, in Chapter 5 we consider the following geometrical problem. Given a smooth manifold M and two smooth vector fields X and Y on M, we want to steer a point $p \in M$ to a point $q \in M$ in minimum time using only integral curves of the two vector fields X and Y. Even if there exists such a trajectory going from p to q, it may happen that the minimum time is obtained only by a trajectory γ whose velocity $\dot{\gamma}(t)$ belongs to the segment joining $X(\gamma(t))$ and $Y(\gamma(t))$ (not being an extremum of it). Thus we first convexify the set of velocities letting $V(x) = \{vX(x) + (1-v)Y(x) : 0 \le v \le 1\}$ be the set of admissible velocities at a generic point x of M. If we fix the point p and let q vary on M we define the function $\mathbf{T}_p(q)$ as the minimum time to reach the point q from p. When M is two dimensional and p is a stable point, that is $0 \in V(x)$, this is precisely the minimum time problem for a system of type (1) (setting $X = \frac{F-G}{2}, Y = \frac{F+G}{2}$). We are interested in investigating the generic properties of the function $\mathbf{T}_p$.

Even if the space of all admissible trajectories is infinite dimensional, we know that that the time optimal trajectories are indeed finite concatenations of X trajectories, Y trajectories and singular trajectories. Moreover, it is known that a function that is the minimum of a finite number of smooth functions in generic position is a Morse function in topological sense [2, 29]. We give a positive answer to the question of V.I. Arnold: is the function $\mathbf{T}_p$ generically a Morse function in topological sense? This result is in the same spirit of [2, 29], being now the minimization taken over a finite dimensional set of functions. In order to prove our result, we analyze the Minimum Time Fronts (briefly MTF) that are the level sets of the minimum time function $\mathbf{T}_p$. The properties of the MTFs are given in Theorem 25. One can define the extremal front in an obvious way. It is interesting to notice that the extremal front at some singular points of the optimal one, generically develops only standard singularities as two dimensional wave front, see [3, 4]. Moreover, these singularities correspond exactly to projection singularities, of the set of lifted trajectories, of cusp type discussed in Chapter 2.

The generic properties of the minimum time front on points reached by abnormal extremals (that is trajectories with vanishing Hamiltonian) are studied. By a local analysis we show that at these points the minimum time front is always tangent to the abnormal extremals. This fact was remarked also in [8] and in [41] for a quadratic cost.

Finally, we study the case in which $M = \mathbb{R}^2$ and p is replaced by a smooth two dimensional manifold with boundary. In this case, under a local controllability assumption, we can also prove the semiconcavity of the minimum time function. This property is typically encountered in calculus of variation problems or optimal control problems with set of velocities $V(x)$ of the same dimension of the state space, see [21] and references therein. (Notice that, under generic conditions, semiconcavity fail with p a point.)

In Chapter 6 there is a simple application. We study some controlled dynamics appearing in Lagrangian systems of mathematical physics. In particular we study the following:

Problem: *Consider the autonomous ODE in* $\mathbb{R}$*:*

$$\begin{array}{rcl} \ddot{y} & = & f(y, \dot{y}), \\ f \in \mathcal{C}^\infty(\mathbb{R}^2), & & f(0,0) = 0 \end{array} \tag{2}$$

that describes the motion of a point under the action of a force depending on the position and velocity of the point (for instance due to a magnetic field or the viscosity of a fluid). Let us apply a external force, that we suppose bounded (e.g. $|u| \leq 1$*):*

$$\ddot{y} = f(y, \dot{y}) + u.$$

Controlling the external force, we want to reach a point (y_0, v_0) *of the configuration space, from the rest state* $(0,0)$*, in minimum time.*

It turns out that the synthesis describing the solutions to this problem may be classified assigning an ordered couple of elements of the set $\mathbb{N} \cup \{\infty\}$. Applications to the Duffin and Van der Pol equations are given.

In Chapter 7 we deal with a different problem in which we use some of the techniques developed in the previous Chapters. We study the stability of a bidimensional switched systems for arbitrary switching functions.

By a switched system we mean a family of continuous–time dynamical systems and a rule that determines at any time which dynamical system is responsible of the time evolution. More precisely let $\{f_u : \ u \in U\}$ be a (finite or infinite) set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{3}$$

The rule is given assigning the so called switching function $u(.) : [0, \infty[\to U$. Here we consider the situation is which the switching function cannot be predicted a priori; it is given from outside and represents some phenomena (e.g. a disturbance) that it is not possible to control or include in the dynamical system model.

Suppose now that all the f_u have a given property for every $u \in U$. A typical problem is to study under which conditions this property holds for the system (3) for arbitrary switching functions. For a discussion of various issues related to switched system we refer the reader to [28].

In [1, 27] the case of switched linear systems was considered:

$$\dot{x} = A_u x, \quad x \in \mathbb{R}^n, \quad A_u \in \mathbb{R}^{n \times n}, \quad u \in U, \tag{4}$$

and the problem of the asymptotic stability of the origin for arbitrary switching functions was investigated. Clearly we need the asymptotic stability of each single sub–system $\dot{x} = A_u x$, in order to have the asymptotic stability of (4) for each switching function (i.e. the eigenvalues of each matrix A_u must have strictly negative real part). This will be assumed to be the case. In [1, 27], it is shown that the structure of the Lie algebra generated by the matrices A_u:

$$\mathbf{g} = \{A_u : \ u \in U\}_{L.A.},$$

is crucial for the stability of the system (4) (i.e. the interrelation between the systems). The main result of [27] is the following. If $\mathbf{g}$ is a solvable Lie algebra, then the switched system (4) is asymptotically stable for each switching function $u(.) : [0, \infty[\to U$.

In [1] a generalization was given. Let $\mathbf{g} = \mathbf{r} \oplus \mathbf{s}$ be the Levi decomposition of $\mathbf{g}$ in its radical (i.e. the maximal solvable ideal of $\mathbf{g}$) and a semi–simple sub–algebra, where the symbol $\oplus$ indicates the semidirect sum. The generalization is the following. If $\mathbf{s}$ is a compact Lie algebra then the system (4) is asymptotically stable for every switching function $u(.) : [0, \infty[\to U$. Anyway the converse of this result is not true in general: if $\mathbf{s}$ is non compact, the system can be stable or unstable. If $\mathbf{s}$ is non compact then it contains as a sub–algebra $sl(2, \mathbb{R})$ and thanks to this fact the authors were able to reduce the problem of the asymptotic stability of the system (4) (when $\mathbf{s}$ is not compact) to the problem of the asymptotic stability of an auxiliary bidimensional system. We refer the reader to [1] for details. For this reason the bidimensional problem assumes a particular interest: we give the complete description of that case for a single input system.

More precisely we study the stability of the origin for the switched system:

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{5}$$

where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part, $x \in \mathbb{R}^2$ and $u(.) : [0, \infty[\to [0, 1]$ is an arbitrary measurable switching function. Moreover we suppose that at least A or B has complex eigenvalues and that they are diagonalizable. We find a necessary and sufficient condition under which the origin of the system (5) is asymptotically stable for each function $u(.)$. Our result is obtained with a direct method without looking for a common Lyapunov function, but analyzing the locus in which the two vector fields are collinear, similarly to what done in Chapter 2 and 3. We make also use of the concept of feedback. For the difficulty of finding a common Lyapunov function for this problem we refer the reader to [23].

Notice that if we set $F(x) = (Ax+Bx)/2$ and $G(x) = (Ax-Bx)/2$ then the switching system (5) has exactly the form (1) with $u \to 2u-1$ and $G(0) = 0$.

Two cases are analyzed separately. The first in which both the matrices have complex eigenvalues (in the following **(CC)** case) and the second in which one of the two matrices has real and the other has complex eigenvalues (in the following **(RC)** case). The case in which both matrices have real eigenvalues is similar and it is treated in [10]. There are only three relevant parameters: one depends on the eigenvalues of A, one on the eigenvalues of B, and the last contains the interrelation between the two systems and it is the cross ratio of the four eigenvectors of A and B in the projective line $\mathbf{C}P^1$.

Fixed the value of the cross ratio we study the asymptotic stability domain. In the **(CC)** case it is formed by one or two open unbounded convex regions. While in the **(RC)** case it is an open unbounded region not always convex.

Chapter 1

Basic Definitions

In this Chapter we recall the notations of [17, 31, 32] and we give the basic definition used in all Chapters 2, 3, 5, 6. We refer to the problem of stabilizing in minimum time to the origin the control system:

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad |u| \leq 1 \tag{1.1}$$

where we assume $F(0) = 0$.

Given a set A we indicate by $\sharp(A)$ its cardinality. For any function $f : A \to B$, A, B sets, the domain is indicated by $Dom(f)$. If X is a topological space and M a subset of X then we indicate by $Clos(M)$ the closure of M in X. A subset A of X is said to be generic if it contains an open and dense set. Analogously a property P is said generic if the set of points satisfying P is generic. If M is a manifold then $dim(M)$ indicates its dimension.

Definition 1 *We say that a vector field is $\mathcal{C}^\infty$ if its components admit partial derivatives of any order that are bounded on the whole plane.*

For every couple of $\mathcal{C}^\infty$ vector fields (F, G) we consider the minimum time stabilization problem to the origin for the control system (1.1). Reversing time we can consider the equivalent problem of reaching every point of the plane in minimum time from the origin.

Given a measurable function $u : [a, b] \to [-1, 1]$, a trajectory of (1.1) corresponding to u is an absolutely continuous map $\gamma : [a, b] \to \mathbb{R}^2$ such that $\dot{\gamma}(x) = F(\gamma(t)) + u(t)G(\gamma(t))$ for almost every $t \in [a, b]$. Since the system is autonomous we can always assume that $Dom(\gamma) = [0, a]$ for some $a \in \mathbb{R}$. A trajectory $\gamma : [0, a] \to \mathbb{R}^2$ is (time) optimal if for every trajectory $\gamma' : [0, b] \to \mathbb{R}^2$ with $\gamma(a) = \gamma'(b)$ we have $a \leq b$. A trajectory γ corresponding to a constant control ± 1 is called a *bang arc*. A *bang-bang* trajectory is a finite concatenation of bang arcs and a time where the control changes sign is called *switching time*.

The well known Pontryagin Maximum Principle in this special case states the following. Define for every $(x, p, u) \in \mathbb{R}^2 \times (\mathbb{R}^2)_* \times [-1, 1]$, where $(\mathbb{R}^2)_*$ is the set of row vectors:

$$\mathcal{H}(x, p, u) = p \cdot F(x) + u\, p \cdot G(x)$$

and:

$$H(x, p) = \max\{p \cdot F(x) + u\, p \cdot G(x) : u \in [-1, 1]\}. \tag{1.2}$$

If $\gamma : [0, a] \to \mathbb{R}^2$ is a (time) optimal trajectory corresponding to a control $u : [0, a] \to [-1, 1]$, then there exist *a nontrivial field of covectors along* γ, that is a function $\lambda : [0, a] \to (\mathbb{R}^2)_*$ never vanishing, and a constant $\lambda_0 \leq 0$ such that:

i) $\dot{\lambda}(t) = -\lambda(t) \cdot (\nabla F + u(t)\nabla G)(\gamma(t))$,

ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$ for almost every $t \in Dom(\gamma)$,

iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$ for almost every $t \in Dom(\gamma)$.

In this case we say that the pair (γ, λ) is extremal. Moreover, if $\lambda_0 = 0$ then we say that (γ, λ) is an abnormal extremal. If γ is optimal we say that the pair (γ, λ) is optimal.

Notice that with the definition 1.2 we can write the equations for (γ, λ) in the Hamiltonian form:

$$\begin{cases} \dot{x} = \partial_p H(x, p) \\ \dot{p} = -\partial_x H(x, p). \end{cases}$$

However H is not a smooth function.

Definition 2 *A stratification of $\mathbb{R}^2$ is a locally finite partition $\mathcal{P}$ of $\mathbb{R}^2$ into connected embedded submanifolds M_i, $i \in \mathbb{N}$, called strata, satisfying the following. Let $M_1, M_2 \in \mathcal{P}$, then*

$$\textit{If } Clos(M_1) \cap M_2 \neq \emptyset, \textit{ then } M_2 \subset Clos(M_1) \textit{ and } dim(M_2) < dim(M_1). \tag{1.3}$$

Definition 3 *A regular optimal synthesis for (1.1) is a collection of trajectory-control pairs $\{(\gamma_x, u_x) : x \in \mathbb{R}^2\}$ satisfying the following properties:*

1. *For every $x \in \mathbb{R}^2$, $\gamma_x : [0, t_x] \to \mathbb{R}^2$ steers the origin to x in minimum time.*

2. *If $y = \gamma_x(t)$ for some $t \in Dom(\gamma_x)$ then γ_y is the restriction to $[0, t]$ of γ_x.*

3. *There exists a stratification of $\mathbb{R}^2$ such that $u(x) = u_x(t_x)$ is smooth on each stratum.*

Let Ξ be the set of all couples of $\mathcal{C}^\infty$ vector fields (F,G) such that $F(0)=0$. From now on we endow Ξ with the the $\mathcal{C}^3$ topology, that is the topology induced by the norm:

$$\begin{aligned}\|(F,G)\| &= \sup\Big\{ \left|\frac{\partial^{\alpha_1+\alpha_2}F_i(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right|, \left|\frac{\partial^{\alpha_1+\alpha_2}G_i(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right| : \ x\in\mathbb{R}^2; \\ &\qquad i=1,2;\ \alpha_1,\alpha_2\in\mathbf{N}\cup\{0\}; \alpha_1+\alpha_2\le 3\Big\}.\end{aligned}$$

In [17, 31], it was proved that there exists a (essentially unique) regular synthesis for every couple of vector fields in an open dense subset of Ξ. Moreover, the control $u(x)$ is constantly equal to $+1$ or to -1 on every two dimensional stratum. On each one dimensional stratum it is equal to $+1$, to -1 or to the feedback control:

$$\varphi(x) = -\frac{\nabla\Delta_B(x)\cdot F(x)}{\nabla\Delta_B(x)\cdot G(x)}, \tag{1.4}$$

where:

$$\Delta_B(x) = G(x)\wedge[F,G](x) = G_1(x)[F,G]_2(x) - G_2(x)[F,G]_1(x), \tag{1.5}$$

and $[F,G]$ is the Lie bracket of F and G. The feedback φ gives the sliding motion on $\Delta_B^{-1}(0)$. If γ corresponds to the feedback φ of (1.4) then γ is called a *singular arc*. For later use we also define the function:

$$\Delta_A(x) = F(x)\wedge G(x) = F_1(x)G_2(x) - F_2(x)G_1(x). \tag{1.6}$$

For convenience we set $Y=F+G$ and $X=F-G$. Let $\gamma:[t_1,t_2]\to\mathbb{R}^2$ be a trajectory of (1.1).

- If γ correspond to constant control $+1$ (resp. -1) in $[t_1,t_2]$ we say that $\gamma|_{[t_1,t_2]}$ is a Y–trajectory (resp. X–trajectory) and we write $\gamma|_{[t_1,t_2]}\in$ Traj(Y) (resp. $\gamma|_{[t_1,t_2]}\in$ Traj(X)).
- If γ correspond to the control $\varphi(\gamma(x))$ in $[t_1,t_2]$ (defined in (1.4)) we say that $\gamma|_{[t_1,t_2]}$ is a Z–trajectory and we write $\gamma|_{[t_1,t_2]}\in$ Traj(Z).

Moreover if $\gamma_1:[t_1,t_2]\to\mathbb{R}^2$, $\gamma_2:[t_2,t_3]\to\mathbb{R}^2$ $(t_1<t_2<t_3)$ are two trajectories of (1.1) with $\gamma_1(t_2)=\gamma_2(t_2)$ we set:

$$(\gamma_2*\gamma_1)(t) := \begin{cases}\gamma_1(t) \text{ for } t\in\ [t_1,t_2[\\ \gamma_2(t) \text{ for } t\in\ [t_2,t_3].\end{cases}$$

Given an extremal trajectory γ we denote by $n(\gamma)$ the smallest integer such that there exist $\gamma_i\in$ Traj$(X)\cup$Traj$(Y)\cup$Traj(Z), $i=1,...,n(\gamma)$ satisfying:

$$\gamma = \gamma_{n(\gamma)} * \dots * \gamma_1. \tag{1.7}$$

The function $n(\gamma)$ is called the number of arcs of γ. In [31] it was shown that under generic conditions each extremal trajectory γ can always be written in the form (1.7) with $n(\gamma) < \infty$ and each arc is bang or singular. These generic conditions are the following:

(P1) The vectors $G(0)$ and $[F, G](0)$ are linearly independent.
(P2) For every $x \in \mathbb{R}^2$, 0 is a regular value for Δ_A and Δ_B.
(P3) The set $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)$ is locally finite.

Let Tan_A be the set of points $x \in \Delta_A^{-1}(0)$ such that $X(x)$ or $Y(x)$ is tangent to $\Delta_A^{-1}(0)$. Define Tan_B in the same way using Δ_B rather than Δ_A.

(P4) Tan_A and Tan_B are locally finite sets.

Let $Bad := (\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)) \cup Tan_A \cup Tan_B$.

(P5) Bad is locally finite.

Notice that **(P5)** is a consequence of **(P3)** and **(P4)**.

(P6) If $x \in Bad,\quad G(x) = 0$ then $F(x) \cdot \nabla(\Delta_A)(x) \neq 0$ and $F(x) \cdot \nabla(\Delta_B)(x) \neq 0$.
(P7) If $x \in Bad,\quad G(x) \neq 0, x \in (\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)) \cap Tan_A$, say $X(x) \cdot \nabla\Delta_A(x) = 0$, then $\partial_y(X \cdot \nabla\Delta_A)|_{y=x} \neq 0$, $X(y) \neq 0$, $Y(y) \neq 0$, where y takes values on $\Delta_A^{-1}(0)$ in a neighborhood of x.
(P8) If $x \in Bad,\quad x \in Tan_B$ then $\Delta_A(x) \neq 0$.
(P9) If $x \in Bad,\quad G(x) \neq 0$, $X(x) = 0$ or $Y(x) = 0$ then $\Delta_B(x) \neq 0$.

The generic conditions **(P6), (P7), (P8)** are showed in the following figure:

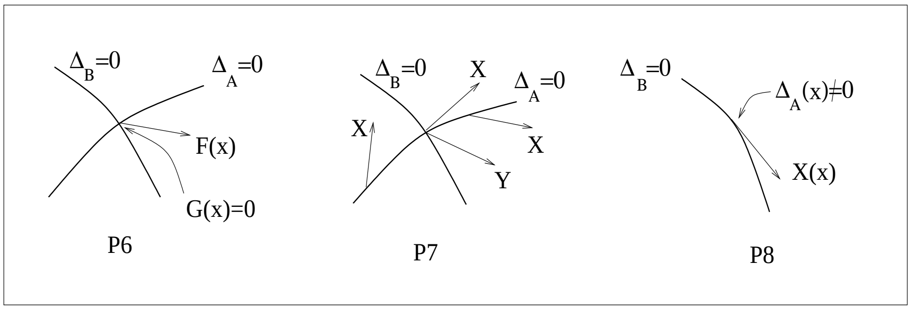


Figure 1.1

Let $\tau > 0$. We call *reachable set* within time τ the set $\mathcal{R}(\tau) := \{x \in \mathbb{R}^2$: there exists $t \in [0, \tau]$ and a trajectory $\gamma : [0, \tau] \to \mathbb{R}^2$ of (1.1) such that $\gamma(0) = 0$, $\gamma(t) = x\}$. In the following we consider a generic τ in the sense that we assume the generic condition:

(GAτ) Let $(\gamma,\lambda):[0,\tau]\to\mathbb{R}^2\times\mathbb{R}_2$ belong to Γ. Then the condition $\phi(\tau)=\lambda(\tau)\cdot G(\gamma(\tau))=0$ implies the following:

- $\tau\notin\{t_i^{\pm},t_i'^{\pm},s_i^{\pm},s_i'^{\pm}\}$ (see Definition 5);
- $\gamma(\tau)\notin\Delta_A^{-1}(0)$. The fact that this is generic follows from Theorem 13 and 3;
- $\gamma(\tau)\in\Delta_B^{-1}(0)\Rightarrow\ \exists\,\varepsilon>0$ such that γ correspond to the singular control φ in $[\tau-\varepsilon,\tau]$ and $|\varphi(\gamma(\tau))|<1$;
- $X(\gamma(\tau))$ and $Y(\gamma(\tau))$ are not tangent to C or $\bar{C}$ curves (see Section 2.1.1).

We call $\gamma^{\pm}:[0,t_f^{\pm}]\to\mathbb{R}^2$ the extremal trajectories exiting the origin with constant control ±1, where $t_f^{\pm}$ are the last times in which $\gamma^{\pm}$ are extremal (if they are less than τ) or τ (otherwise).

Definition 4 *Let $(\gamma,\lambda):[0,\tau]\to\mathbb{R}^2$ be an extremal pair. The corresponding switching function is defined as $\phi(t):=\lambda(t)\cdot G(\gamma(t))$.*

We have the following:

Proposition 1 *Let $(\gamma,\lambda):[0,\tau]\to\mathbb{R}^2$ be an extremal pair and $[t_1,t_2]\subseteq[0,\tau]$ $(t_1<t_2)$. Then we have the following.*

- *On $[t_1,t_2]$ γ corresponds to constant control +1 (resp. -1) iff for each $t\in[t_1,t_2]$ we have $\phi(t)\geq0$ (resp. $\phi(t)\leq0$) and $meas(\{t\in[t_1,t_2]:\ \phi(t)=0\})=0$.*
- *On $[t_1,t_2]$ γ corresponds to the singular control φ iff for each $t\in[t_1,t_2]$ we have $\phi(t)=0$.*

Proposition 2 *Let γ be an extremal trajectory and $\bar{t}\in Dom(\gamma)$ a switching time. Let $\bar{x}=\gamma(\bar{t})$ and suppose that $\bar{x}\notin\Delta_A^{-1}(0)\cup\Delta_B^{-1}(0)$. Then $\bar{t}$ is a switching time from X to Y iff $-\Delta_A(\bar{x})/\Delta_B(\bar{x})>0$ and $\bar{t}$ is a switching time from Y to X iff $-\Delta_A(\bar{x})/\Delta_B(\bar{x})<0$.*

We say that an open connected subset S of $\Delta_B^{-1}(0)$ is a *turnpike* (resp. *antiturnpike*) if for each $x\in S$ we have the following: **(i)** $X(x)$ and $Y(x)$ are not tangent to S and point to opposite side; **(ii)** $\Delta_A(x)\neq0$; **(iii)** there exists a small neighborhood U of x such that $U\cap\Delta_A^{-1}(0)=\emptyset$, $U\setminus\Delta_B^{-1}(0)$ has two connected components, $-\Delta_B(x)/\Delta_A(x)>0$ (resp. $-\Delta_B(x)/\Delta_A(x)<0$) on the component into which $X(x)$ points and $-\Delta_B(x)/\Delta_A(x)<0$ (resp. $-\Delta_B(x)/\Delta_A(x)>0$) on the components into which $Y(x)$ points. Now, if an extremal pair (γ,λ) verifies $\gamma(t)=x$, $x\in\Delta_B^{-1}(0)$, $\lambda(t)\cdot G(\gamma(t))=0$ and $\Delta_B^{-1}(0)$ is a turnpike in a neighborhood of x then there exist $\varepsilon>0$ and an extremal trajectory defined on $[0,t+\varepsilon]$ coinciding with γ up to time t and corresponding to the feedback control φ on $[t,t+\varepsilon]$. That is we can enter turnpikes with an extremal trajectory at a switching time. Moreover, if there exists an extremal trajectory γ, a time $\bar{t}>0$ and $\varepsilon_1,\varepsilon_2>0$ such that γ is a bang arc in $[\bar{t}-\varepsilon_1,\bar{t}]$ and a singular arc in $[\bar{t},\bar{t}+\varepsilon_2]$, then $Supp(\gamma|_{[\bar{t},\bar{t}+\varepsilon_2]})\subset\Delta_B^{-1}(0)$. The following proposition shows that in fact $Supp(\gamma|_{[\bar{t},\bar{t}+\varepsilon_2]})$ must be a *turnpike*.

Proposition 3 *Let (γ,λ) be an extremal pair such that $\gamma(\bar{t}) \in \Delta_B^{-1}(0) \setminus \Delta_A^{-1}(0)$ for some $\bar{t} \in Dom(\gamma)$. Set $\bar{x} := \gamma(\bar{t})$ and suppose that $X(\bar{x})$ and $Y(\bar{x})$ point to opposite side of $\Delta_B^{-1}(0)$. Let U be a small neighborhood of $\bar{x}$ such that $U \cap \Delta_A^{-1}(0) = \emptyset$ and $U \setminus \Delta_B^{-1}(0)$ has two connected components. Call these components U_X and U_Y labelled in such a way $X(\bar{x})$ points in U_X. Suppose that $-\Delta_B(\bar{x})/\Delta_A(\bar{x}) > 0$ on U_X and $-\Delta_B(\bar{x})/\Delta_A(\bar{x}) < 0$ on U_Y. Then $\lambda(\bar{t}) \cdot G(\gamma(\bar{t})) \neq 0$.*

Proof. First suppose that γ reaches x from U with bang control, say $+1$, so that γ enters U_X. The function $\phi(t)$ (see Definition 4) satisfies the equation $\dot{\phi}(t) = \lambda(t) \cdot [F,G](\gamma(t))$. Since $U \cap \Delta_A^{-1}(0) = \emptyset$, we can write for $x \in U$, $[F,G](x) = f(x)F(x) + g(x)G(x)$ where $f(x) = -\Delta_B(x)/\Delta_A(x)$ and $g(x)$ is a smooth function (see [31]). It follows $\dot{\phi}(t) = f(\gamma(t))\lambda(t) \cdot F(\gamma(t)) + g(\gamma(t))\lambda(t) \cdot G(\gamma(t))$ for $t \leq \bar{t}$. Now if γ is an abnormal extremal then the condition $\lambda(\bar{t}) \cdot G(\gamma(\bar{t})) = 0$ would imply $\gamma(\bar{t}) \in \Delta_A^{-1}(0)$ (see Chapter 3), so we may assume

$$\lambda(t) \cdot (F(\gamma(t)) + G(\gamma(t))) = cost > 0 \tag{1.8}$$

for every $t \in Dom(\gamma)$. Assume by contradiction that $\lambda(\bar{t}) \cdot G(\gamma(\bar{t})) = 0$. From (1.8), taking $t' \leq \bar{t}$ sufficiently close to $\bar{t}$ we may assume $\lambda(t) \cdot F(\gamma(t)) > 0$ for every $t \in [t', \bar{t}]$. Now for $t \in [t', \bar{t}[$ we have $f(\gamma(t)) > 0$, hence if k is such that $|g(x)| < k$ for $x \in U$, we get

$$\begin{aligned} &\dot{\phi}(t) = f(\gamma(t))\lambda(t) \cdot F(\gamma(t)) + g(\gamma(t))\lambda(t) \cdot G(\gamma(t)) > -k\phi(t) \quad \forall t \in [t', \bar{t}], \\ &\phi(t') = \phi_0 > 0, \end{aligned}$$

hence ϕ is strictly positive on $[t', \bar{t}]$, reaching a contradiction. Therefore the only possibility is that γ enters U with singular control φ. This can happen only if there exists $\tilde{t} < \bar{t}$ and $\varepsilon > 0$ such that γ corresponds to the singular control φ in $[\tilde{t} - \varepsilon, \tilde{t} + \varepsilon]$ and $\gamma(\tilde{t}) \in \Delta_B^{-1}(0) \cap \Delta_A^{-1}(0)$ but this is excluded by the following arguments. If $F(\gamma(\tilde{t})) + \varphi(\gamma(\tilde{t}))G(\gamma(\tilde{t})) = 0$ then $\gamma(\tilde{t})$ can not be reached in finite time along $\Delta_B^{-1}(0)$. Assume now that $F(\gamma(\tilde{t})) + \varphi(\gamma(\tilde{t}))G(\gamma(\tilde{t})) \neq 0$. If $G(\gamma(\tilde{t})) = 0$ then by **(P6)** $\gamma|_{[\tilde{t}-\varepsilon,\tilde{t}]}$ can not be on the set $\Delta_B^{-1}(0)$. Finally the case $G(\gamma(\tilde{t})) \neq 0$ is excluded by (**P8**). ■

In [17, 31, 32] the one and zero dimensional strata (submanifold of $\mathbb{R}^2$) of the optimal synthesis are called respectively *Frame Curves* and *Frame Points* (in the following briefly FCs and FPs). Moreover, the authors used the letters X, Y, C, S, K, respectively to indicate the FCs corresponding to subset of $\gamma^+([0, t_f^+])$, subsets of $\gamma^-([0, t_f^-])$, curve made of switching points (switching curves), turnpikes and overlaps, that are curves made of points reached optimally by two distinct trajectories. In [32], it was proved that these are generically all possible FCs. While a FP x that is intersection of two FCs F_1 and F_2 is called a (F_1, F_2) Frame Point.

1.1 Extremals Bifurcating from $\gamma^+ \cup \gamma^-$

The optimal synthesis in a neighborhood of $Supp(\gamma^+) \cup Supp\,(\gamma^-)$ for the control problem (1.1) was constructed in [17]. The extremal one is obtained here in a very similar way. Given an extremal trajectory γ, let us define $v^\gamma(v_0, t_0; t)$ to be the solution to the Cauchy problem:

$$\begin{cases} \dot{v^\gamma}(v_0, t_0; t) & = & (\nabla F + u(t)\nabla G)(\gamma(t)) \cdot v^\gamma(v_0, t_0; t) \\ v^\gamma(v_0, t_0; t_0) & = & v_0, \end{cases} \tag{1.9}$$

where $u(.)$ is the control corresponding to γ. In [31], denoting $\bar{v}^\gamma(t) := v^\gamma(G(\gamma(t)), t; 0)$ it was defined the function

$$\theta^\gamma : \;\; Dom(\gamma) \to \;[-\pi, \pi], \;\; \theta^\gamma(t) := \; arg(\bar{v}^\gamma(0), \bar{v}^\gamma(t)), \tag{1.10}$$

where arg is the angle measured counterclockwise. We obviously have the following:

Proposition 4 *For fixed t_0, t, the map $f_{t_0,t} : v_0 \mapsto v^\gamma(v_0, t_0; t)$ is linear and injective. Moreover let $\gamma : [a, b] \to \mathbb{R}^2$ be an extremal trajectory of (1.1), corresponding to a constant control $\bar{u}$. Then for every $t, t_0 \in [a, b]$ it holds: $v^\gamma((F+\bar{u}G)(\gamma(t_0)), t_0; t) = (F+\bar{u}G)(\gamma(t))$.*

In the following we use the notation $\theta^\pm(t) := \theta^{\gamma^\pm}(t)$ and $v^\pm := v^{\gamma^\pm}$. From Proposition 4 we have: $v^\pm((F \pm G)(\gamma^\pm(t)), t; 0) = (F \pm G)(0) = \pm G(0)$.

In [17, 31] it was shown:

Proposition 5 *We have the following:*

(A) $t_f^+ = \begin{cases} \tau \textit{ if } |\theta^+(a) - \theta^+(b)| < \pi \;\; \forall \;\; a, \; b \in \; [0, \tau]; \\ \min\{t \in [0, \tau] : \;\; |\theta^+(a) - \theta^+(b)| = \pi \textit{ for some } a, \; b \in \; [0, t]\} \textit{ otherwise}; \end{cases}$

(B) *let γ be an extremal trajectory, then for almost every $t \in Dom(\gamma)$ we have $sgn(\dot{\theta}^\gamma(t)) = sgn(\Delta_B(\gamma(t))$;*

(C) *let γ be an extremal trajectory, then for each $t \in \; Dom(\gamma)$ the conditions $\phi(t) = \lambda(t) \cdot G(\gamma(t)) = 0, \;\; \dot{\theta}^\gamma(t) \neq 0$ imply that t is a switching time for γ.*

Moreover the following times were single out:

Definition 5

$$
\begin{aligned}
&t_0^+ = t_0'^+ = 0\\
&t_1^+ = \inf\{t > 0: \quad \theta^+ \textit{ has a local max. at } t, \quad \theta^+(t) > 0\}\\
&t_i^+ = \inf\{t > t_{i-1}: \quad \theta^+ \textit{ has a local max. at } t, \quad \theta^+(t) > \theta^+(t_{i-1}^+)\}\\
&t_1'^+ = \inf\{t > 0: \quad \theta^+ \textit{ has a local min. at } t, \quad \theta^+(t) < 0\}\\
&t_i'^+ = \inf\{t > t_{i-1}': \quad \theta^+ \textit{ has a local min. at } t, \quad \theta^+(t) < \theta^+(t_{i-1}^+)\}\\
&s_i^+ = \begin{cases} \max\{t \in [t_{i-1}^+, t_i^+]: \quad \theta^+(t) = \theta^+(t_{i-1}^+)\} \textit{ if } t_i^+ \textit{ is defined,}\\ \max\{t \in [t_{i-1}^+, t_f^+]: \quad \theta^+(t) = \theta^+(t_{i-1}^+)\} \textit{ otherwise}\end{cases}\\
&s_i'^+ = \begin{cases} \max\{t \in [t_{i-1}'^+, t_i'^+]: \quad \theta^+(t) = \theta^+(t_{i-1}'^+)\} \textit{ if } t_i'^+ \textit{ is defined,}\\ \max\{t \in [t_{i-1}'^+, t_f'^+]: \quad \theta^+(t) = \theta^+(t_{i-1}'^+)\} \textit{ otherwise.}\end{cases}
\end{aligned}
$$

Similarly the times t_i^-, $t_i'^-$, s_i^-, $s_i'^-$, were defined.

Observation **1** We can have three situations (cfr. Figure 1.2):

1) $|\theta^\pm(a) - \theta^\pm(b)| < \pi$ for every $a, b \in [0, \tau]$. In this case $t_f^\pm = \tau$ and $|\theta^\pm(t_f^\pm)| < \pi$;

2) $|\theta^\pm(a) - \theta^\pm(b)| = \pi$ for some $a, b \in]0, \tau]$. In this case $|\theta^\pm(t_f^\pm)| < \pi$, at least $t_1^\pm$ or $t_1'^\pm$ are defined and either $s_1^\pm \neq 0$ or $s_1'^\pm \neq 0$

3) $|\theta^\pm(t_f^\pm)| = \pi$. In this case, from **(GAτ)** we get $t_f^\pm < \tau$.

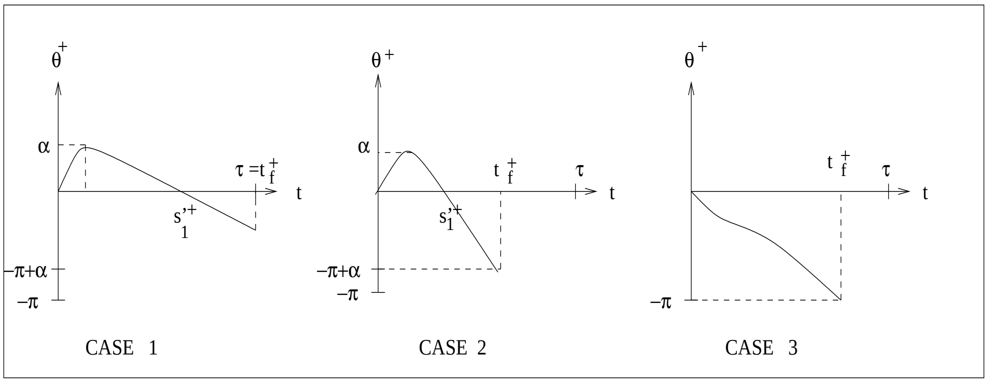


Figure 1.2

The structure of the extremal synthesis near $Supp(\gamma^+)$ is contained in the following Theorem that include Propositions 3.1, 3.2, 3.3 of [17]. The only difference is that here it is written in terms of the FCs $\bar{C}$ instead of the FCs K. The Theorem holds under the following generic assumptions (called **SA1÷SA9** in [17]):

(GA1) For every $t \in [0, t_f^+]$, $G(\gamma^+(t)) \neq 0$.
(GA2) $\dot{\theta}^+(0) \neq 0$, $\dot{\theta}^+(t_f^+) \neq 0$.

(GA3) If $\dot{\theta}^+(t) = 0$, then $\theta^+(t) \neq 0$, $\ddot{\theta}^+(t) \neq 0$.
(GA4) If $t \neq s$ and $\dot{\theta}^+(s) = \dot{\theta}^+(t) = 0$ then $\theta^+(s) \neq \theta^+(t)$.
(GA5) $(\nabla\Delta_B \cdot V)(\gamma^+(t)) \neq 0$, $V = X, Y$ at all points $t \in \{t_i,\ t_i' :\ \ i \geq 1\}$.
(GA6) If $t_f^+ = \tau$ then $\max\{|\theta^+(t) - \theta^+(\tau)|,\ \ t \in [0, \tau]\} < \pi$. Notice that **(GA6)** is a consequence of **(GAτ)**.
(GA7) Let v be the unit tangent vector at $\gamma^+(s_i^+)$ to the switching curve starting at $\gamma^+(s_i^+)$. We have $v \cdot X(\gamma^+(s_i^+)) \neq 0$, for each $i \geq 2$. And similarly for the points $\gamma^+(s_i'^+)$.
(GA8) Suppose $s_1'^+ \neq 0$ and consider a local system of coordinates in a neighborhood of $\gamma^+(s_1'^+)$ such that $Y = (1, 0)$, $X(0) = (-1, 0)$, $X(s_1, 0) = (a, 0)$ and define $b := -\partial_{x_1} X_2(0)$, $c := \partial_{x_1} X_2(s_1, 0)$. We have $ac - \frac{1}{2}b(1-a)^2 \neq 0$.
(GA9) $\gamma^+(t) \neq (0, 0)$ for every $t > 0$.

Theorem 1 *Let the conditions* **(P1)**–**(P9)**, **(GA1)**–**(GA9)** *holds. Then*

- *if t^* lies in one of the open intervals $]s_i^+, t_i^+[$ or $]s_i'^+,\ t_i'^+[$ then there exists an extremal control of the form:*

$$u(t) = \begin{cases} 1 & \text{if} \quad t \in\ [0, t^*[, \\ -1 & \text{if} \quad t \in\ [t^*, t^* + \varepsilon[, \end{cases} \tag{1.11}$$

  for some $\varepsilon > 0$. In the first case the trajectory bifurcates to the right of γ^+ (clockwise), in the second case to the left (counterclockwise). On the other hand, no extremal control of the form (1.11) exists if t^* is not contained in any one of the closed intervals $[s_i^+, t_i^+]$ or $[s_i'^+,\ t_i'^+]$;

- *to the right of each point $\gamma^+(t_i^+)$ ($t_i^+ > 0$) and to the left of each point $\gamma^+(t_i'^+)$ ($t_i'^+ > 0$) originates either a turnpike, when the inner products $\nabla\Delta_B \cdot X$ and $\nabla\Delta_B \cdot Y$ have opposite signs, that is when the two vector fields X, Y points to opposite sides of the set of zeros of Δ_B, or a switching curve of conjugates points, when the signs are equal;*

- *if $t_f^+ < \tau$ then a switching curve of kind C or $\bar{C}$ originates from $\gamma^+(t_f^+)$. Such curve bifurcates to the left of γ^+ when $\dot{\theta}^+(t_f^+) > 0$, and to the right of γ^+ when $\dot{\theta}^+(t_f^+) < 0$;*

- *to the right of each points $\gamma^+(s_i^+)$ ($i \geq 2$) and to the left of each point $\gamma^+(s_i'^+)$ ($i \geq 2$) the extremal synthesis contains a switching curve of kind C or $\bar{C}$;*

- *if $s_1^+ > 0$ (resp. $s_1'^+ > 0$) then to the right of $\gamma^+(s_1^+)$ (resp. to the left of $\gamma^+(s_1'^+)$) the extremal synthesis contains a switching curve C or $\bar{C}$.*

A similar result holds for γ^- under similar generic assumptions. More details about the points $\gamma^+(s_1^+)$ (if $s_1^+ \neq 0$), $\gamma^+(s_1'^+)$ (if $s_1'^+ \neq 0$) and $\gamma^+(t_f^+)$ (in the case $|\theta^+(t_f^+)| = \pi$) in which X and Y are collinear, are given in Chapter 3.

Chapter 2

Extremal Synthesis

In this Chapter we consider all the extremal trajectories for the minimum time stabilization to the origin of the control system:

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad |u| \leq 1, \quad F(0) = 0. \tag{2.1}$$

We construct extremal syntheses and we classify all their singularities and all the singularities of the projection of the set of support of normalized extremals from $\mathbb{R}^2 \times S^1$ to $\mathbb{R}^2$.

2.1 Preliminary Facts and Statement of the Main Results

In [17] an algorithm was defined to construct, under suitable conditions, a structurally stable regular optimal synthesis. Roughly speaking the algorithm constructs at step n the extremal arcs made of n pieces each of which is either a bang arc or a singular arc. The not optimal arcs are cut by the algorithm. We now describe a new algorithm to construct a regular extremal synthesis. First we need some definitions.

Definition 6 *Let M be a manifold (embedded in $\mathbb{R}^n$) and $\Lambda \in M$ a set such that $\Lambda = \cup_{j\in J} M_j$, where $J \subset \mathbb{N}$ and M_j are disjoint embedded connected submanifolds of $\mathbb{R}^n$. For any $z \in \Lambda$ a neighborhood of z in Λ is given by $U \cap \left(\cup_{j\in J(z)} M_j\right)$ with U neighborhood of z in M and $j \in J(z)$ if and only if $z \in Clos(M_j)$. Then Λ is a Withney stratified set if the collection $\mathcal{P} := \{M_j\}_{j\in J}$, called the stratification of Λ, is locally finite in Λ, (1.3) is satisfied and the following holds.*

- *Let $x_n, y_n \in M_j$, $n \in \mathbb{N}$, $x_n, y_n \to \bar{x} \in M_k \subset Clos(M_j)$ and denote by ℓ_n the straight line in $\mathbb{R}^n$ containing the segment joining x_n with y_n. If $T_{x_n} M_j \to T$ (affine subspace of $\mathbb{R}^n$) and $\ell_n \to \ell$, then $\ell \subset T$ and $T_{\bar{x}} M_k \subset T$.*

We define the dimension of Λ by $dim(\Lambda) = \max_j dim(M_j)$ and say that Λ is regular if every $z \in \Lambda$ belongs to the closure of a stratum of dimension $dim(\Lambda)$. If the collection $\mathcal{P} = \{M_j\}_{j\in J}$ is locally finite in M then we say that Λ is locally finite in M.

Given an extremal pair (γ, λ) one easily checks that for every $\alpha \in \mathbb{R}$, $\alpha > 0$, the pair $(\gamma, \alpha\lambda)$ is also extremal. It is then natural to introduce the following projection:

$$\pi_* : \mathbb{R}^2 \times \mathbb{R}^2_* \setminus \{0\} \to \mathbb{R}^2 \times S^1 \qquad \pi_*(x,p) = \left(x, \frac{p}{|p|}\right). \tag{2.2}$$

A key role is played by the trajectory-covector pairs $(\gamma^{\pm}, \lambda^{\pm})$ starting from the origin and corresponding to constant control ± 1.

Definition 7 *A regular extremal synthesis Γ^* for (2.1) is a collection of trajectory-covector pair sets $\{\Gamma^*_{(x,p)} : (x,p) \in T^*(\mathbb{R}^2) = \mathbb{R}^4\}$ satisfying the following properties:*

1. *For every $(x,p) \in \mathbb{R}^4$, $\Gamma^*_{(x,p)}$ is the collection of all trajectory-covector extremal pairs $(\gamma, \lambda) : [0,a] \to \mathbb{R}^4$ such that $\gamma(0) = 0$, $\gamma(a) = x$ and $\lambda(a) = p$.*

2. *For every $(x,p) \in \mathbb{R}^4$ and $\bar{a} \in \mathbb{R}$, $\bar{a} \geq 0$, the set $\{(\gamma,\lambda) : (\gamma,\lambda) \in \Gamma^*_{(x,p)}, Dom(\gamma) \subset [0,\bar{a}]\}$ is finite. For every $x \in \mathbb{R}^2$, $x \notin \gamma^{\pm}(Dom(\gamma^{\pm}))$, and $\bar{a} \in \mathbb{R}$, $\bar{a} \geq 0$, the set $\{\pi_*(\gamma,\lambda) : \ \gamma(\bar{a}) = x, \ \ (\gamma,\lambda) \text{ extremal}\}$ is finite.*

3. *For every $(x,p) \in \mathbb{R}^4$ and every $(\gamma,\lambda) \in \Gamma^*_{(x,p)}$, γ is a finite concatenation of bang and singular arcs.*

4. *The set $\mathcal{N} = \{(x,p) : \Gamma^*_{(x,p)} \neq \emptyset\}$ is a regular Withney stratified subset of $\mathbb{R}^4$ and $dim(\mathcal{N}) = 3$.*

5. *The set of discontinuities of $\#(\Gamma^*_{(x,p)})$ on $\mathcal{N}$ (i.e. where $\mathcal{N}$ fails to be a manifold) is a countable collection of two dimensional embedded submanifolds of $\mathbb{R}^4$ (precisely a countable collection of strata of the stratification $\mathcal{P}$ of $\mathcal{N}$.)*

The algorithm constructs first a local regular extremal synthesis in a neighborhood of the extremal trajectory-covector pairs $(\gamma^{\pm}, \lambda^{\pm})$. The set of extremal pairs is subdivided into "strips". Each strip is a one dimensional parametric family of extremal pairs. The strips evolution is then studied. In order to do that, the evolution of the boundaries of strips is analyzed separately. The more delicate case is that of abnormal extremals. Moreover the evolution of the interior of a strip may create new strips and cause the subdivision of each strip in smaller strips. In Section 2.1 the local synthesis near $(\gamma^{\pm}, \lambda^{\pm})$ is studied. In Section 2.2 the subdivision in strips is operated and a detailed description of the algorithm is given. In Section 2.3 the case of abnormal extremals is treated. Finally in Section 2.4 the evolution of interior of strips will complete the analysis.

For every $\tau > 0$ the algorithm, under generic assumptions on (F, G), constructs in a finite number of steps all the extremal pairs defined on the interval $[0, \tau]$. Under generic assumptions on τ (see Chapter 1.1, Section 1.1, condition **(GAτ)**), the collection of these pairs has all the properties required by the definition of regular extremal synthesis with the set of discontinuities of $\#(\Gamma^*_{(x,p)})$ being finite. Moreover, the set $\mathcal{N}$ (restricted to time τ) is a locally closed and locally finite (in $\mathbb{R}^4$) regular Withney stratified set. Using generic conditions similar to those of [17], we are able to prove structural stability (see Section 2.5 for precise definition) of the extremal synthesis for a.e. fixed time. For the difficulties arising in extending structural stability at infinite time, we refer the reader to [22]. We thus obtain the following:

Theorem 2 *There exists an open dense subset Π of Ξ such that for every $(F, G) \in \Pi$ there exists a regular extremal synthesis Γ^*. Moreover for almost every fixed time τ, Γ^* is structurally stable.*

As a byproduct we obtain a detailed study of the singularities for the extremal regular synthesis. There exists a finite set of (equivalence classes of) stable singularities described by Theorems 10 and 11 (see Section 2.6).

For each singular point $z \in \mathbb{R}^4$ around which $\mathcal{N}$ is a manifold, we describe the corresponding topological singularity of the projection of the extremal synthesis to the base $\mathbb{R}^2$. More precisely we normalize covectors obtaining the set $N := \pi_*(\mathcal{N}) \subset \mathbb{R}^2 \times S^1$ ($\mathcal{N}$ as in Definition 7). Since N is locally a two dimensional piecewise regular manifold around z, as singularities of its projection on $\mathbb{R}^2$ we can expect to find the topological counterparts of stable singularities of maps between two dimensional manifolds, namely folds and cusps. In fact we have two new stable singularities as shown by the theorem below. A bifold is roughly speaking obtained folding two times a sheet, while a ribbon is more complicated (see definition 9 and figure 2.20).

Definition 8 *A stable topological singularity of the projection of an extremal regular synthesis is a stable topological singularity for the projection of $N = \pi_*(\mathcal{N})$ (see Definition 7) onto $\mathbb{R}^2$ outside a neighborhood of the set of discontinuities of $\#(\Gamma^*_{(x,p)})$ (where $\mathcal{N}$ fails to be a manifold).*

Theorem 3 *Under generic assumptions on $(F, G) \in \Xi$, each stable topological singularity of the projection of the extremal regular synthesis (given by Theorem 2) is of one of the following type (see Section 2.1 and Section 2.6 for definitions and figures):*

1. *Vertical (normal, origin, cusp, fold, ending)*

2. *Fold,*

3. *Cusp,*

4. Bifold,

5. Ribbon.

Only 1.,2.,3. can happen on a small neighborhood of the intersection with the set of optimal pairs, with fold points not being optimally reached. Moreover 5. appears only along abnormal extremals.

2.1.1 New Frame Curves

In [17, 31, 32] (see also Chapter 1) the one and zero dimensional strata (submanifold of $\mathbb{R}^2$) of the optimal synthesis are called respectively *Frame Curves* and *Frame Points* (in the following briefly FCs and FPs). As we will see in next Sections, N (Recall that we set $N := \pi_*(\mathcal{N})$) is a stratified subset of $\mathbb{R}^2 \times \mathcal{S}^1$ of dimension two. With some abuse of notation we call also the one and zero dimensional strata of N Frame Curves and Frame Points respectively. Moreover during the construction of the extremal synthesis we deal with the projection of the curves X, Y, C, S, K, and their intersections (i.e. FPs) on $\mathbb{R}^2$. We use the same name for the curves and points in $\mathbb{R}^2 \times \mathcal{S}^1$ and for those projections on $\mathbb{R}^2$, being from the context clear if we are treating the former or the latter.

In the following Sections we construct the extremal synthesis. In the construction of the optimal synthesis at each step some extremal non optimal trajectories are cut by the algorithm generating the so called overlap curves. Now we keep all extremal trajectories thus the overlap curves are not constructed and some new FCs appears. In the following we illustrate the main differences with respect to the optimal case.

- we do not have any K (overlap) FC because these FCs are obtained as "cuts" of not optimal extremal trajectories (see [17, 31] and Chapter 5);
- Beside the C curves on which X and Y point to the same side, we also have a new type of FC (called $\bar{C}$) on which X and Y point to opposite sides (see figure 2.1 Case 1 and 2).

By definition switching curves are loci of conjugate points to other FCs and $\bar{C}$ was excluded from the optimal synthesis because trajectories switching on $\bar{C}$ are not locally optimal. Notice that a C FC can be smoothly joint to a $\bar{C}$ FC if the switching locus becomes tangent to X or Y. Other FCs that appear building the extremal synthesis are:

- a FC called W that is an arc of an extremal trajectory characterized by the fact that the projection of all the extremal trajectories near to it lye on the same side (see figure 2.1 Case 3).

- A FC called γ_0 that is an arc of an extremal trajectory that "transports" some special information, e.g. it switches every time it meets the locus $\Delta_A^{-1}(0)$ or it evolves into a W FC. The optimal synthesis near a γ_0 FC is regular, but possibly the covectors do not depend in a smooth way from points on the plane.

More details are given later.

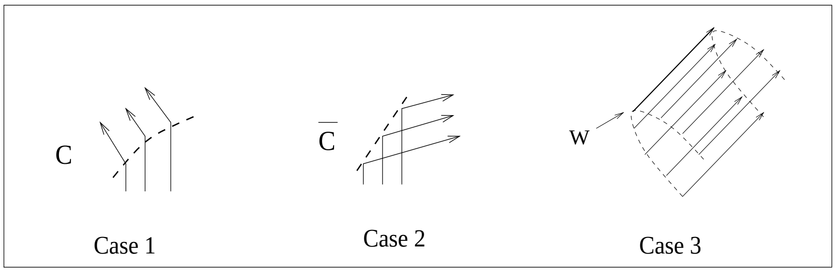


Figure 2.1: FCs of kind C, $\bar{C}$ and W

2.1.2 Singularities along $\gamma^+ \cup \gamma^-$

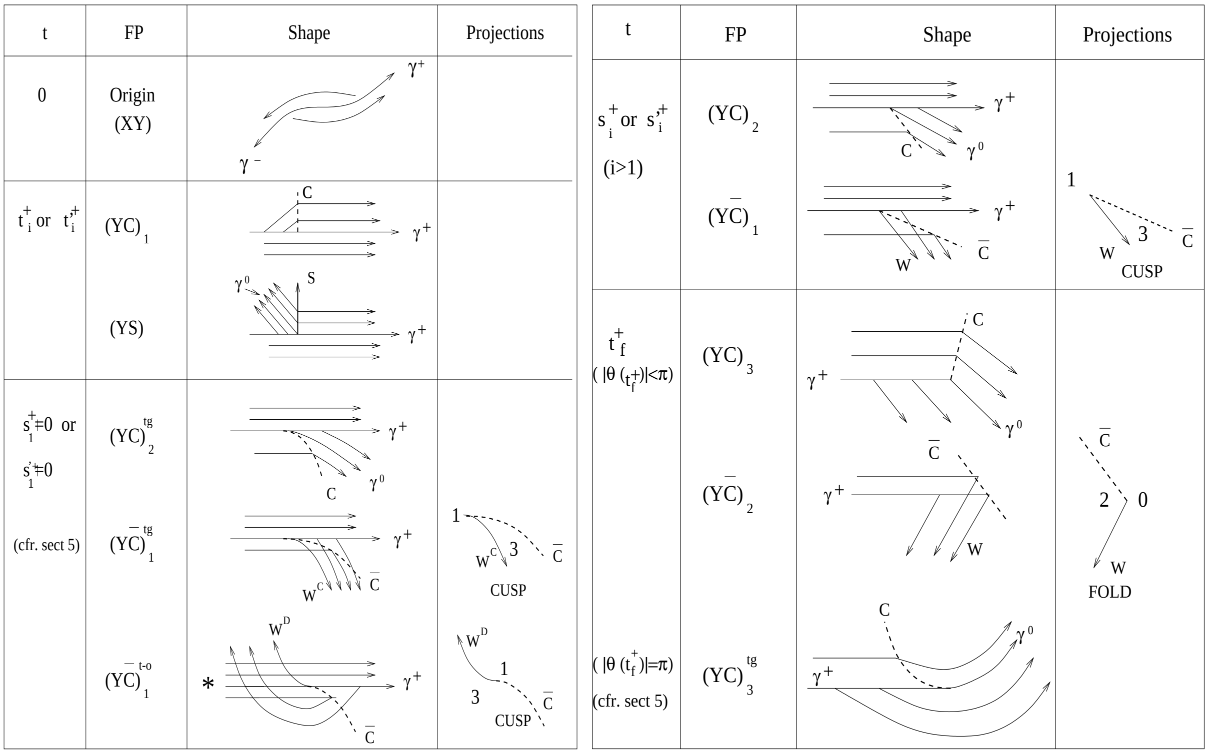


Figure 2.2 and 2.3: FPs along $\gamma^+ \cup \gamma^-$. In the point labelled with $*$ the trajectories self intersect

The FPs of the optimal synthesis along $\gamma^+ \cup \gamma^-$ are described in [32] (see also figure 5.1, Chapter 5). For the extremal synthesis case we have the same FPs, but the extremal synthesis does not contain any K curve and, on the other side may present $\bar{C}$, γ_0, W. The local extremal synthesis around these points is depicted in figures 2.2 and 2.3. Let x be such a FP. In figure 2.2 and 2.3 the first column indicates the time t such that $x = \gamma^+(t)$, the second shows the type of Frame Point, the third depicts the synthesis near x, finally the fourth indicates the corresponding projection singularity (see details below). The superscripts "tg" and "t-o" denotes respectively the fact that the two FCs are tangent with the same, resp. opposite, versus. The superscripts "C" and "D" on the FCs W are explained in Section 2.3.1. Moreover at the points labelled with $*$ (e.g. $(Y\bar{C})_1^{t-o}$) the trajectories self intersect but not there lift in the cotangent bundle. The situation is studied in Chapter 4. The singularities along γ^- are similar.

2.1.3 Number of pre–images

From the analysis in a neighborhood of $Supp(\gamma^+) \cup Supp(\gamma^-)$, it is clear that if we projects the extremal synthesis on $\mathcal{R}(\tau)$, there can be subsets of $\mathcal{R}(\tau)$ covered by more than one trajectory. In fact the borders of these subsets correspond to singularities of the projection from N and they are treated below and in more details in [16].

Fix $\bar{x} \in \mathcal{R}(\tau)$ and an extremal trajectory $\bar{\gamma} : [0, \bar{a}] \to \mathcal{R}(\tau)$ such that $\bar{\gamma}(0) = 0$ and $\bar{\gamma}(\bar{a}) = \bar{x}$ and define the function:

$$\begin{aligned} K_\varepsilon^{\bar{x},\bar{\gamma}}(x) \;:=\; & \sharp\{\text{extremal trajectories } \gamma : \quad \gamma(0) = 0,\ \gamma(a) = x,\ |\gamma(t) - \bar{\gamma}(t)| < \varepsilon \\ & \forall\, t \in\ [0, min(a, \bar{a})],\quad |a - \bar{a}| < \varepsilon\}. \end{aligned} \tag{2.3}$$

We refer to figure 2.4 where for simplicity we drop the dependence on $\bar{x}$, $\bar{\gamma}$ and ε.

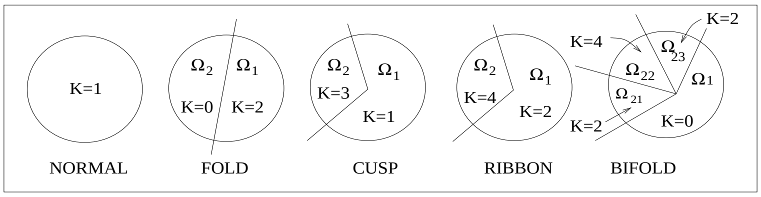


Figure 2.4

Definition 9 *Fix $\bar{x} \in \mathcal{R}(\tau)$ and an extremal trajectory $\bar{\gamma} : [0, \bar{a}] \to \mathcal{R}(\tau)$ such that $\bar{\gamma}(0) = 0$ and $\bar{\gamma}(\bar{a}) = \bar{x}$;*

- *we say that $\bar{x}$ is a* normal point *along $\bar{\gamma}$ if for ε sufficiently small there exists a neighborhood U of $\bar{x}$ such that $K_\varepsilon^{\bar{x},\bar{\gamma}}(U) = 1$;*

- *we say that $\bar{x}$ is a* fold point *along $\bar{\gamma}$ if there exists a one dimensional piecewise–$\mathcal{C}^1$ manifold l, with $\bar{x} \in l$, satisfying the following. For ε sufficiently small there exists a neighborhood U of $\bar{x}$ divided by l into two connected components Ω_1, Ω_2 such that $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1) = 0$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_2) = 2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(l) = 1$;*

- *if l, $U(\varepsilon)$, Ω_1, Ω_2 are as above then if $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1) = 1$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_2) = 3$ and $K_\varepsilon^{\bar{x},\bar{\gamma}}(l) = 2$, we say that $\bar{x}$ is a* cusp point *along $\bar{\gamma}$,*

- *if l, $U(\varepsilon)$, Ω_1, Ω_2 are as above then if $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1) = 2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_2) = 4$ and $K_\varepsilon^{\bar{x},\bar{\gamma}}(l) = 3$, we say that $\bar{x}$ is a* ribbon point *along $\bar{\gamma}$,*

- *we say that $\bar{x}$ is a* bifold point *along $\bar{\gamma}$ if the there exists a one dimensional connected piecewise–$\mathcal{C}^1$ embedded manifold l and two connected $\mathcal{C}^1$ embedded manifolds l_1 and l_2 satisfying the following:*

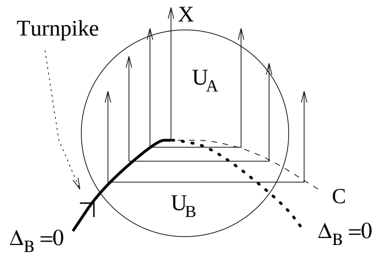
  - *$l \cap \partial l_i = \{x\}$ $(i = 1, 2)$; $\partial l_1 \cap \partial l_2 = \{x\}$;*
  - *for ε sufficiently small there exists a neighborhood U of x satisfying the following. The set $U \setminus l$ has two connected components Ω_1, Ω_2 and the set $\Omega_2 \setminus \{l_1 \cap l_2\}$ has three connected components Ω_{21}, Ω_{22}, and Ω_{23} (the name are chosen as in figure 2.4) such that $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_1) = 0$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_{21}) = 2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_{22}) = 4$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(\Omega_{23}) = 2$, $K_\varepsilon^{\bar{x},\bar{\gamma}}(l) = 1$ and $K_\varepsilon^{\bar{x},\bar{\gamma}}(l_i) = 3$ $(i = 1, 2)$.*

Example of ribbon and bifold points are given in Section 2.3. Notice that the manifolds l, l_1, l_2 where $K_\varepsilon^{\bar{x},\bar{\gamma}}$ is discontinuous correspond to FCs of kind $\bar{C}$ and W.

2.1.4 The $(SC)_2$ singularity

After the singularities along $Supp(\gamma^+) \cup Supp(\gamma^-)$ the first singularities we need to study are the end points of turnpikes. These singularities where studied in [32] and called (SC_2) points (see also Chapter 5). One has the following

Theorem 4 *Consider an extremal trajectory γ corresponding to a constant control (say +1) in $]t_0 - \varepsilon, t_0]$ and to the singular control φ in $]t_0, t_0 + \varepsilon[$. Then the trajectory $\tilde{\gamma}$ satisfying $\tilde{\gamma}|_{[0,t_0+\varepsilon[} = \gamma|_{[0,t_0+\varepsilon[}$ and corresponding to the singular control φ after $t_0 + \varepsilon$ is extremal exactly up to the first time in which X or Y becomes tangent to the set of zeros of $\Delta_B^{-1}(0)$. Let $\bar{t}_0$ be this time, U a small neighborhood of $x_0 := \gamma(\bar{t}_0)$ and U_A, U_B the two connected components of $U \setminus \Delta_B^{-1}(0)$ chosen in such a way that $X(x_0)$ and $Y(x_0)$ point in U_A. Assume $Y(x_0) \cdot \nabla\Delta_B^{-1}(0) = 0$. Then a switching curve of kind C originates at x_0 and it lies in U_A (see figure below).*



2.2 Strips and the Algorithm

Now we give the key definition of *strip*. A strip is essentially a one parameter continuous family of extremal bang–bang trajectories having the same switching strategy.

Definition 10 *Let a, b be two real numbers s.t. $0 \le a < b \le \tau$ and $x \in \mathcal{R}(\tau)$. A set of trajectories $\mathcal{S}^{a,b,x} = \{\gamma_\alpha : \quad \alpha \in [a,b],\ \gamma_a(a) = x\}$ is called a strip if:*

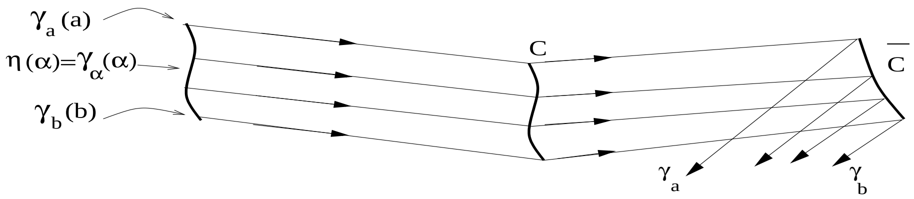
i) $\forall\ \alpha \in [a,b], \quad \gamma_\alpha : [0, \tau(\alpha)] \to \mathbb{R}^2,\ \tau(\alpha) > \alpha$ *is an extremal trajectory for the control problem (2.1). Moreover there exists $\varepsilon > 0$ s. t. $\gamma_{[\alpha,\alpha+\varepsilon]}$ corresponds to a constant control ± 1;*

ii) $\forall\ \alpha \in]a,b[, \quad \gamma_\alpha$ *does not switch on $\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$ after time α;*

iii) *The set $\mathcal{B}^{a,b,x} = \{y \in \mathcal{R}(\tau) : \quad \exists\ \alpha \in\ ]a,b[$ and $t \in]\alpha, \tau(\alpha)[$ s. t. $y = \gamma_\alpha(t)$, t is a switching time for $\gamma_\alpha\}$ is never tangent to X or Y.*

iv) *The map $\eta : \ \alpha \in\ [a,b] \mapsto \gamma_\alpha(\alpha) \in \mathbb{R}^2$ is a bang or singular arc.*

The function $\eta : [a,b] \to \mathbb{R}^2$ is called the *base* of the strip, $\overset{\circ}{\mathcal{S}}{}^{a,b,x} := \{\gamma_\alpha : \quad \alpha \in]a,b[\}$ is called an open strip and $\partial\mathcal{S}^{a,b,x} := \{\gamma_a,\ \gamma_b\}$ is called *strip border*.



Since trajectories generate from both side of a turnpike, one gets immediately:

Proposition 6 *If $\mathcal{S}^{a,b,x}$ is a strip and its base η is a singular arc then there exists a strip $\mathcal{S}' \neq \mathcal{S}^{a,b,x}$ with the same base (the two strips start from the two sides of the turnpike).*

Now we define an algorithm that constructs all the extremal trajectories up to time τ.

<u>ALGORITHM</u>

STEP 1 We construct all the open strips $\overset{\circ}{\mathcal{S}}^{a,b,x}$ with $x \in \; Supp(\gamma^+) \cup \; Supp(\gamma^-)$ and maximal base. In particular these strips are the strips which bases are $\gamma^+|_{]s_i^+,t_i^+[}$, $\gamma^+|_{]s_i'^+,t_i'^+[}$, $\gamma^+|_{]s_i^-,t_i^-[}$, $\gamma^+|_{]s_i'^-,t_i'^-[}$ and the strips obtained in the following way. Suppose that at time t_i^+ (resp. $t_i'^+$, t_i^-, $t_i'^-$) a turnpike starts and let γ be the trajectory corresponding to the constant control +1 (resp. +1, -1, -1) in the interval $[0,t_i^+]$ (resp. $[0,t_i'^+]$, $[0,t_i^-]$, $[0,t_i'^-]$) and to the singular control φ in $]t_i^+,b]$ (resp. $]t_i'^+,b]$, $]t_i^-,b]$, $]t_i'^-,b]$) where $b=\tau$ (if the turnpike does not reach a $(SC)_2$ singular point before τ) or b is the time in which the turnpike meets a $(SC)_2$ singular point (see Theorem 4). In this case we have also the two strips having $\gamma|_{]t_i^+,b[}$ (resp. $\gamma|_{]t_i'^+,b[}$, $\gamma|_{]t_i^-,b[}$, $\gamma|_{]t_i'^-,b[}$) as base (cfr. Proposition 6). Notice that the strips built in this way are a finite number.

STEP 2 We extend every trajectory of all the strips built in the previous step(s) up to the maximal time in which **ii)** and **iii)** of Definition 10 are satisfied.

STEP 3 If all the trajectories built in the previous steps satisfy $\tau(\alpha)=\tau$ we jump to step 4 otherwise:

step 3.1 let γ_α be a trajectory built in the previous steps that satisfies $\tau(\alpha)<\tau$ (that means that **ii)** or **iii)** are violated at $\gamma_\alpha(\tau(\alpha))$) and suppose that this trajectory belongs to the open strip $\overset{\circ}{\mathcal{S}}^{a,b,x}$ ($\alpha \in]a,b[$). Then we substitute this strip with the two open strips $\overset{\circ}{\mathcal{S}}^{a,\alpha,x}$ and $\overset{\circ}{\mathcal{S}}^{\alpha,b,x}$;

step 3.2 if at $\gamma_\alpha(\tau(\alpha))$ a turnpike starts then we build also the two new open strips (with maximal base) having this turnpike as base;

step 3.3 jump to step 2.

STEP 4 We construct all the borders of the strips built in the previous steps.

Notice that a border never corresponds to the singular control φ. Moreover from the conditions **(P1)**–**(P9)**, **(GA1)**–**(GA9)**, **(GA**τ**)** and from the analysis of next Sections it follows that the Algorithm stops after a finite number of steps. Indeed from [31] we have that under the generic conditions **(P1)**–**(P9)**, all extremals are finite concatenations (with an uniform bound on the number of arcs) of bang and singular arcs. The number of strips is also finite, because under generic conditions, the assumptions **ii)** and **iii)** of Definition 10 are violated in a finite number of points. In order to construct a stable extremal synthesis we assume various generic conditions on FCs and FPs during the execution of the algorithm. Some of these conditions are explicitly stated in [17] Section 3, and others are given in the following Sections, namely conditions **(GA10)**–**(GA16)**. After the algorithm ended we need some more generic stability conditions. In [17] it was defined an equivalence among Frame Points, e introduce here a similar one:

Definition 11 *If $x_1,\ x_2$ are two FPs, we say that they are equivalent if there exist some points $y_0 = x_1,\ y_2, ..., y_n = x_2$ such that the following holds. Each y_i belongs to a frame curve D_i. For every y_i, $i = 1, ..., n-1$, there exist an extremal trajectory γ_i and $a_i,\ b_i \in Dom(\gamma_i)$, satisfying $\gamma_i(a_i) = y_i,\ \gamma_i(b_i) = y_{i+1}, \gamma_i|_{[a_i,b_i]}$ is an X or Y trajectory and $\gamma_i|_{[a_i,b_i]}$ is not part of $\gamma^{\pm}$, γ^0 and W FCs. That is there exists a curve connecting x_1 with x_2 formed by X and Y arcs of extremal trajectories and that no one of these arcs is part of $\gamma^{\pm}$, γ^0, W FCs.*

To understand the meaning of this equivalence see the corresponding definition given in Section 3 of [17].

It is possible to give natural stability conditions for each FP (in Section 3 of [17] they are stated the stability conditions for FPs of the optimal synthesis). We assume the following:

($\mathcal{A}$) all FPs are stable. If $x_1,\ x_2$ are two equivalent FPs then $x_1 = x_2$.

Confront with the condition ($\mathcal{A}1$) and ($\mathcal{A}2$) of Section 3 of [17]. To ensure stability, we need the following:

($\mathcal{F}$) the inequality $\phi(t) \geq 0$ of the first part of Proposition 1 is verified in strict sense on the interior of each interval where an extremal trajectory corresponds to a constant bang control.

This is equivalent to condition ($\mathcal{F}1$) of Section 3 of [17]. Finally, notice that the stability assumption ($\mathcal{F}2$) of [17] is a consequence of **(GAτ)**.

Definition 12 *We denote by Γ the trajectories built by the Algorithm. The trajectories built before step 4 are denoted by $\overset{\circ}{\Gamma}$ and the set of the borders by $\partial\Gamma$.*

Notice that Γ is the canonical projection of Γ^* (see Section 2.1) from $\mathbb{R}_2 \times \mathbb{R}^2$ to $\mathbb{R}^2$. Moreover with the above Definition all the FPs are contained in the base or borders of the strips. Moreover in the interior of a strip we can have only C and $\bar{C}$ one dimensional singularities. This is due to the following:

Theorem 5 *Let $\mathcal{S}^{a,b,x}$ be a strip built by the Algorithm, $\gamma_\alpha \in \overset{\circ}{\mathcal{S}}{}^{a,b,x}$, $\gamma_\alpha :\ [0,\tau] \to \mathcal{R}(\tau)$ and $t_0 \in\ ]\alpha,\tau[$. Then there exist $\varepsilon_1,\ \varepsilon_2,\ \delta$ ($\varepsilon_1 > \delta > 0,\ \varepsilon_2 > 0$) such that for each $\beta \in]\alpha - \varepsilon_2, \alpha + \varepsilon_2[$ all the trajectories γ_β have the same constant control , say +1, in the interval $]t_0 - \varepsilon_1, t_0 - \varepsilon_1 + \delta[$ and one of the following possibilities occur:*

1) *for each $\beta \in]\alpha - \varepsilon_2, \alpha + \varepsilon_2[$, γ_β does not switch in $]t_0 - \varepsilon_1, t_0 + \varepsilon_1[$;*

2) *for each $\beta \in]\alpha - \varepsilon_2, \alpha + \varepsilon_2[$, γ_β switches from Y to X (and there are not other switchings) in $]t_0 - \varepsilon_1, t_0 + \varepsilon_1[$, and the switching locus is of kind C;*

3) *for each* $\beta \in]\alpha-\varepsilon_2, \alpha+\varepsilon_2[$, γ_β *switches from* Y *to* X *(and there are not other switchings) in* $]t_0-\varepsilon_1, t_0+\varepsilon_1[$, *and the switching locus is of kind* $\bar{C}$*;*

Proof. From **ii)** of Definition 10 we have only the following possibilities:

1) t_0 is not a switching time for γ_α,

2) t_0 is a switching time from Y to X for γ_α.

By the generic assumptions **(P1)**–**(P9)**, **(GA1)**–**(GA9)**, **(GA**τ**)** there exists $\varepsilon_1 > 0$ such that:

- γ_α does not switch in $]t_0-\varepsilon_1, t_0+\varepsilon_1[$, in the first case;
- γ_α switches only one time in $]t_0-\varepsilon_1, t_0+\varepsilon_1[$ with a YX switching, in the second case.

By continuity $\Delta_A^{-1}(0)$ and $\Delta_B^{-1}(0)$ never vanishes in a neighborhood of $\gamma_\alpha(t_0)$, hence there exists ε_2 such that for each $\beta \in]\alpha-\varepsilon_2, \alpha+\varepsilon_2[$

- γ_β does not switch in $]t_0-\varepsilon_1, t_0+\varepsilon_1[$ in the first case;
- γ_β switches only one time in $]t_0-\varepsilon_1, t_0+\varepsilon_1[$ (with a YX switching) in the second case.

To conclude the proof it remains to check that the switching locus is entirely of kind C or $\bar{C}$, and this is a consequence of **iii)** of Definition 10. ■

The singularities of the bases were already studied in Section 2.1 except for the starting points of the turnpikes if they do not belong to $Supp(\gamma^+) \cup Supp(\gamma^-)$. To enter a turnpike we need to switch on $\Delta_B^{-1}(0)$, so **ii)** of Definition 10 implies that these singularities happen only on borders of strips.

2.2.1 Strip Borders: the FCs of Kind γ_0 and W

We clearly have:

Proposition 7 $\partial\Gamma$ *is a finite set.*

Moreover, by construction, every border belongs to two different strips and adjacent strips correspond locally to the same control. More precisely it holds:

Proposition 8 *Let* $\gamma \in \partial\Gamma$, $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *be the two stripes such that* $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$ *and* $\bar{t}$ *be a switching time for* γ*. Then the switching loci of* $\mathcal{S}^1$ *and* $\mathcal{S}^2$*, passing trough* $\gamma(\bar{t})$ *correspond both to switchings from* Y *to* X *or both to switchings from* X *to* Y*..*

Definition 13 *Let $\gamma \in \partial\Gamma$ and suppose that it corresponds to a constant control in the interval $]b, c[$ $(0 < b < c \leq \tau)$. We say that in $]b, c[$ γ is a strip border of kind W if for every $t \in\ ]b, c[$, $\gamma(t)$ is a fold point. Viceversa if for every $t \in\ ]b, c[$, $\gamma(t)$ is a normal point, we say that in $]b, c[$ γ is a strip border of kind γ_0.*

By construction we get:

Proposition 9 *If $\gamma \in \partial\Gamma$, γ is of kind W in $]a, b[$ and of kind γ_0 in $]b, c[$ $(0 \leq a < b < c \leq \tau)$ or viceversa then b is a switching time for γ.*

In Section 2.3 we study the singularities of the borders of strips that are *abnormal extremals*. In Section 2.4 we study all the singularities of the other borders.

2.3 Abnormal Extremals

By definition an abnormal extremal is an extremal pair (γ, λ) such that the corresponding Hamiltonian $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = \lambda(t) \cdot (F + u(t)G)(\gamma(t))$ vanishes almost everywhere on $Dom(\gamma)$. In this Section we study the singularities that involve abnormal extremals. In a neighborhood of such singularities, the extremal trajectories may self intersect. A detailed analysis of the self intersections of extremal trajectories is given in Chapter 4. In Figure 2.6, 2.7, 2.8, 2.9 the singularities with self intersections are labelled with $*$. Some proof of this section are postponed to Chapter 3 in which more details about abnormal extremals are given. Section 3.1 of Chapter 2 should be thought as an appendix of this section.

Definition 14 *Let $\gamma : [0, \tau] \to \mathcal{R}(\tau)$ be (the first component of) an abnormal extremal for the control problem (2.1) such that $n(\gamma) \geq 2$ and t_1 its first switching time. We refer to the couples (γ, t_1) as Non Trivial Abnormal Extremal (in the following NTAE). By definition an NTAE is maximal if defined on $[0, \tau]$.*

In Chapter 2 it is proved the following:

Theorem 6 *Let $\gamma : [0, \tau] \to \mathbb{R}^2$ be an extremal trajectory for the control problem (3.1) such that it switches at least one time, $\lambda : [0, \tau] \to \mathbb{R}_2$ the corresponding covector and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Then it holds:*

A) *$\lambda(.)$ is unique (up to the multiplication by a positive constant);*

B) *under generic assumptions the following conditions are equivalent:*

  (a) *(γ, t_1) is a NTAE;*

  (b) *$\gamma(t_i) \in \Delta_A^{-1}(0)$ for some $i \in \{1, ..., n(\gamma) - 1\}$;*

(c) $\gamma(t_i) \in \Delta_A^{-1}(0)$ *for each* $i \in \{1, ..., n(\gamma) - 1\}$*;*

(d) $\gamma(\bar{t}) \in \Delta_A^{-1}(0)$ $(\bar{t} \in Dom(\gamma))$ *iff* $\bar{t} = t_i$ *for some* $i \in \{1, ...n(\gamma) - 1\}$*;*

C) *under generic assumptions,* **a)** *(or equivalently* **(b)** *or* **(c)** *or* **(d)***) implies that for every interval* $[a, b] \subset [0, \tau]$ $(a < b)$*,* γ *does not correspond to the singular control* φ*.*

From Theorem 6 and Definition 10 we have the following:

Corollary 1 *Let* (γ, t_1) *a NTAE, then* $\gamma \in \partial\Gamma$*.*

Hence we have the natural:

Definition 15 $\partial\Gamma_A = \{\gamma \in \partial\Gamma : \ (\gamma, t_1)$ *is a NTAE for some* $t_1 \in \mathbb{R}^+\}$*.*

From now on we assume the following generic condition that guarantee **C)** of Theorem 6:

(GAφ) $\gamma \in \Gamma$ implies that γ does not switch on $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)$.

Definition 16 *Let* $x \in \Delta_A^{-1}(0)$*, then* $X(x) = \alpha Y(x)$*. If* $\alpha > 0$ *(resp.* $\alpha < 0$*) we say that at* x*,* $\Delta_A^{-1}(0)$ *is direct (resp. inverse).*

Let $s_1^{\pm}$, $s_1'^{\pm}$, $t_f^{\pm}$ be the times defined as in Section 1.1. We have (see Theorem 3, Chapter 2):

Proposition 10 *Under generic assumptions there are at most two maximal NTAE. Moreover*

- *one exits the origin with control* $+1$ *and its first switching is at* s_1^+ *iff* $s_1^+ \neq 0$*, or at* $s_1'^+$ *iff* $s_1^+ \neq 0$*, or at* t_f^+ *iff* $|\theta^+(t_f^+)| = \pi$*;*

- *the other exits the origin with control* -1 *and its first switching is at* s_1^- *iff* $s_1^- \neq 0$ *or at* $s_1'^-$ *iff* $s_1'^- \neq 0$ *or at* t^- *iff* $|\theta^-(t_f^-)| = \pi$*.*

Finally if $|\theta^{\pm}(t_f^{\pm})| = \pi$ *then* $\Delta_A^{-1}(0)$ *is direct at* $\gamma(t_f^{\pm})$*.*

2.3.1 Strip borders of kind W^C and W^D

In case of abnormal extremals, we need a more fine definition for the strip borders of kind W.

Definition 17 *We refer to figure 2.5. Let* $\gamma \in \ \partial\Gamma_A$*, and suppose that it corresponds to a constant control (say +1) in the interval* $]b, c[$ $(0 < b < c \leq \tau)$*. Let* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *the two strips such that* $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$ *and suppose that in the interval* $]b, c[$ γ *is a strip border of kind* W*.*

- *We say that in* $]b,c[$ γ *is a strip border of kind* W^C *if* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *lie both to the right (resp. to the left) of* $\gamma|_{]b,c[}$ *and* X *points to the right (resp. to the left) of* $\gamma|_{]b,c[}$ *at every point of* $Supp(\gamma|_{]b,c[})$.
- *We say that in* $]b,c[$ γ *is a strip border of kind* W^D *if* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *lie both to the right (resp. to the left) of* $\gamma|_{]b,c[}$ *and* X *points to the left (resp. to the right) of* $\gamma|_{]b,c[}$ *at every point of* $Supp(\gamma|_{]b,c[})$.

If in the interval $]b,c[$ γ *is not a strip border of kind* W *we say that* $\gamma|_{]b,c[}$ *is a strip border of kind* γ_0.

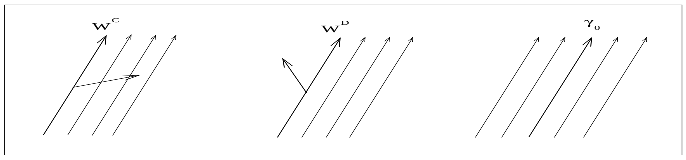


Figure 2.5: The abnormal extremals of type W^C, W^D and γ_0

From Theorem 6 we have the following:

Proposition 11 *Let* $A_1, A_2 \in \{W^C, W^D, \gamma_0\}$ *and* $\gamma \in \partial\Gamma_A$. *If* γ *is of kind* A_1 *in* $]a,b[$ *and of kind* A_2 *in* $]b,c[$ $(A_1 \neq A_2,\ 0 < a < b < c \leq \tau)$ *then* b *is a switching time for* γ

Now it is clear the meaning of this more fine definition in the case of the abnormal extremals. For abnormal extremals a W^C curve can evolve into a W^D curve only at a switching time. Viceversa, in the case of strip borders that are not abnormal extremal this can occur without switching and so the difference between W^C and W^D is not necessary (see Section 2.4).

2.3.2 Singularities

In this subsection we describe all possible Frame Point occurring along a NTAE. By Proposition 10 it follows that the first singularity of $\gamma \in \partial\Gamma_A$ is one of the following (cfr. figure 2.2 and 2.3): $(YC)_2^{tg}$, $(Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}$, $(YC)_3^{tg}$. The subsequent singular points are determined via Proposition 11 and the following Theorem proved in the next Chapter (cfr. Theorem 18)

Theorem 7 *Let* $\gamma \in \partial\Gamma_A$, $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *be two stripes such that* $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$. *Let* $\bar{t}$ *be a switching time for* γ *and* U *a small neighborhood of* $\gamma(\bar{t})$ *such that* $U \setminus \Delta_A^{-1}(0)$ *has two connected components* U_{in} *and* U_{out}, *chosen in such a way* γ *enters* U *from* U_{in}. *Then the switching loci of* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *passing trough* $\gamma(\bar{t})$ *lies both in* U_{in} *or both in* U_{out}; *moreover they are tangent to* $Supp(\gamma)$ *in* $\gamma(\bar{t})$.

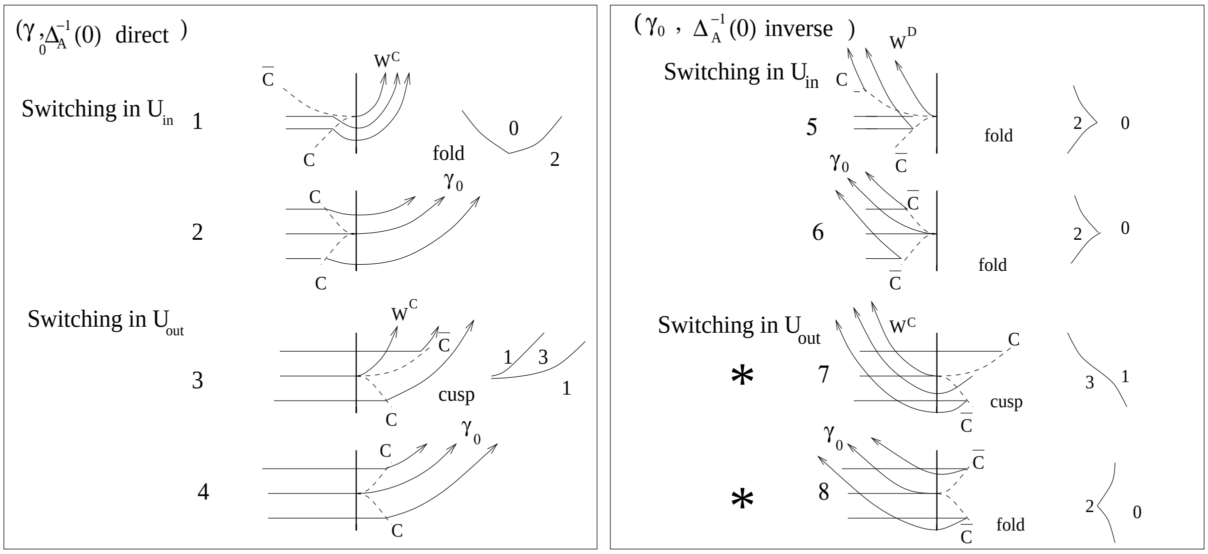
In figures 2.6, 2.7, 2.8, 2.9 are listed all the possible singularities for a NTAE of the kind γ_0, W^C, and W^D that switches on $\Delta_A^{-1}(0)$. These are determined by:

- $\Delta_A^{-1}(0)$ direct or inverse;

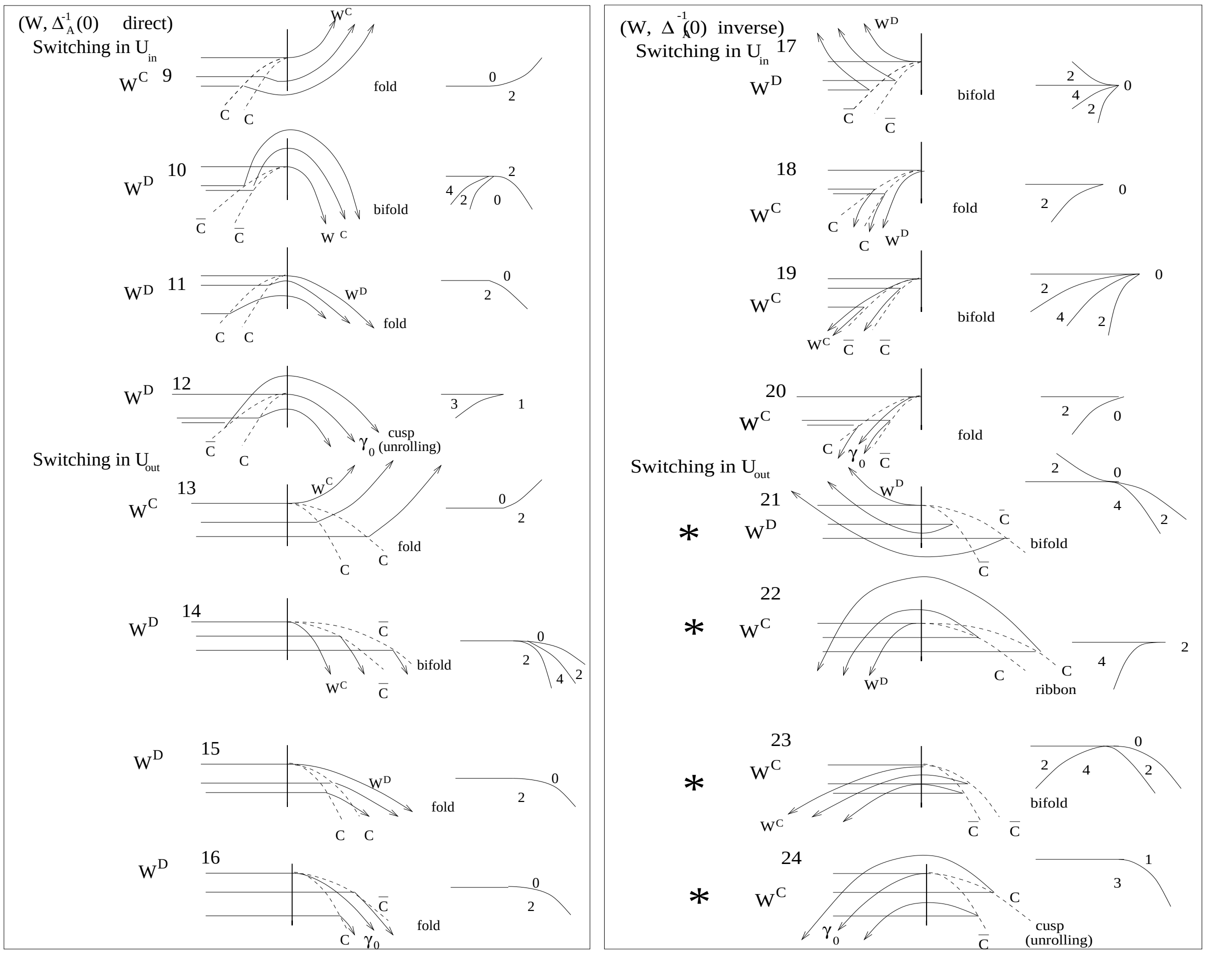
- switching in U_{in} or U_{out} according to Theorem 7;

- all the *essentially different* direction of the exiting abnormal trajectory.

The points labeled with "$*$" are the points in which the trajectories self intersects.



Figures 2.6, 2.7: Singularities for a NTAE of kind γ_0



Figures 2.8, 2.9: Singularities for a NTAE of kind W^C, and W^D

In particular call $\bar{x} := \gamma(\bar{t})$ the singular point and let $I \in \{X, Y\}$ be the vector field such that γ is a I–trajectory in $[\bar{t} - \varepsilon, \bar{t}]$ for some $\varepsilon > 0$. Moreover chose a local system of coordinates in a small neighborhood U of $\bar{x}$ such that $\bar{x} = (0, 0)$, $I = (1, 0)$ and $\Delta_A^{-1}(0) \cap U = \{(x, y) \in \mathbb{R}^2 \cap U : \quad x = 0\}$. In this way (referring to Theorem 7) we have:

$$U_{in} = \{(x, y) \in \mathbb{R}^2 \cap U : \quad x \le 0\}$$
$$U_{out} = \{(x, y) \in \mathbb{R}^2 \cap U : \quad x \ge 0\}.$$

First suppose that γ is a strip border of kind γ_0, $\Delta_A^{-1}(0)$ is direct at $\bar{x}$ and the switching loci of the two strips $\mathcal{S}^1$ and $\mathcal{S}^2$ such that $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$ lie in U_{in}. Under these hypotheses $Supp(\gamma) \cap U_{out}$ has equation:

$$x_2 = a x_1^2 + O(x_1^3) \quad \text{for } x_1 > 0$$

and the switching loci of $\mathcal{S}^1$ and $\mathcal{S}^2$ are represented by the following equation.

$$x_2 = b x_1^2 + O(x_1^3)$$
$$x_2 = c x_1^2 + O(x_1^3)$$
$$b > 0,\ c > 0,\ x < 0.$$

We assume:

(GA10) a, b, c are distinct and different from zero.

Suppose $a > 0$, being the other case similar. If $a > b$ then for some $\varepsilon > 0$ the trajectory $\gamma|_{[\bar{t},\bar{t}+\varepsilon]}$ is a strip border of kind W^C. In this case we say that the singularity if of type 1 (see figure 2.6). On the other hand, if $a < b$ then for some $\varepsilon > 0$ the trajectory $\gamma|_{[\bar{t},\bar{t}+\varepsilon]}$ is a strip border of kind γ_0. In this case we say that the singularity is of type 2 (see figure 2.6). The other cases can be treated similarly under similar generic assumptions and these singularities include the singularities at t_1. From the above assumptions we have the following:

Proposition 12 *Let (γ, t_1) be a NTAE and $t_1 < t_2 < \ldots < t_{n(\gamma)-1}$ the sequence of its switching times. Set $t_0 = 0$ and $t_{n(\gamma)} = \tau$. Then, under generic conditions, for every $i = 1, \ldots, n(\gamma) - 1$, if $\gamma|_{[t_{i-1},t_i]}$ is a strip border of one of the kind W^C, W^D or γ_0, then $\gamma|_{[t_i,t_{i+1}]}$ is also a strip border of one of the kind W^C, W^D or γ_0.*

By induction one gets:

Proposition 13 *Under generic conditions, the singularities 1-24 of figure 2.2 (case $(YC)_2^{tg}$, $(Y\bar{C})_1^{tg}$ $(Y\bar{C})_1^{t-o}$), and 2.3 (case $(YC)_3^{tg}$), 2.6, 2.7, 2.8, 2.9 include all the possible singularities along a NTAE.*

The proof of the fact that all these singularities are in fact realized, is in the next Chapter.

2.4 Strips Evolution and Frame Points

In this Section we complete the study of strips evolution under generic assumptions and classify all possible Frame Points that are generated as well as the new Frame Curves.

For strip borders we have to consider the evolution of γ_0 and W Frame Curves, while for the interior of strips the presence of turnpikes and the violation of condition **iii)** (of Definition 10) generate new Frame Curves and causes the division of strips in smaller strips.

As a byproduct we obtain a complete classification of all possible stable Frame Points of the extremal synthesis.

2.4.1 Frame Curves and Frame Points along C and $\bar{C}$

We first prove a theorem used in the following.

Theorem 8 *Consider an extremal trajectory $\bar{\gamma}$ that switches on $\Delta_B^{-1}(0)$ at time $\bar{t}$. Assume $\Delta_A(\bar{\gamma}(\bar{t})) \neq 0$ and that $X(\bar{\gamma}(\bar{t})) \cdot \Delta_B^{-1}(0)(\bar{\gamma}(\bar{t}))$ and $Y(\bar{\gamma}(\bar{t})) \cdot \Delta_B^{-1}(0)(\bar{\gamma}(\bar{t}))$ are both positive or both negative. Under generic assumptions there exists $\varepsilon > 0$ such that every extremal trajectory γ satisfying $\|\gamma - \bar{\gamma}\|_{C^0} < \varepsilon$ switches before crossing $\Delta_B^{-1}(0)$ iff it switches after crossing $\Delta_B^{-1}(0)$.*

Proof. From the assumptions of the statement we have that the set $\Delta_B^{-1}(0)$ is not a turnpike (indeed is a barrier, see [37]) in a neighborhood of $\bar{\gamma}(\bar{t})$). Moreover, if ε is sufficiently small then every γ satisfying $\|\gamma-\bar{\gamma}\|_{\mathcal{C}^0} < \varepsilon$ crosses the set $\Delta_B^{-1}(0)$ near $\bar{\gamma}(\bar{t})$ and we can define the function $\theta_\gamma(t) = arg(G(\gamma(\bar{t})), v(G(\gamma(t)), t; \bar{t}))$. By definition $\theta_{\bar{\gamma}}(\bar{t}) = 0$ and from Lemma 5 (case B) we have that there exists $\delta > 0$ such that $\theta_{\bar{\gamma}}$ is monotone increasing (decreasing), in $[\bar{t}-\delta, \bar{t}]$ and monotone decreasing (increasing) in $[\bar{t}, \bar{t}+\delta]$. For ε sufficiently small θ_γ has the same behavior that is there exists t_γ such that $|t_\gamma - \bar{t}| < (\delta/4)$, θ_γ is monotone increasing (decreasing), in $[\bar{t}-(3\delta/4), t_\gamma]$ and monotone decreasing (increasing) in $[t_\gamma, \bar{t}+(3\delta/4)]$, and $|\theta_\gamma(\bar{t} \pm (3\delta/4)) - \theta_{\bar{\gamma}}(\bar{t} \pm (3\delta/4))| < (|\theta_{\bar{\gamma}}(\bar{t} \pm (3\delta/4))|/4)$. (Notice that $t(\gamma)$ is precisely the time at which γ crosses the set $\Delta_B^{-1}(0)$.) Assume now that γ switches at a point s_γ near $\bar{t}$ then $\lambda_\gamma(s_\gamma) \cdot G(\gamma(s_\gamma)) = 0$ (here λ_γ denotes the covector associated to γ). Generically, $s(\gamma) \neq t(\gamma)$ and if ε is sufficiently small then $|\theta_\gamma(s_\gamma)| < \min\{|\theta_\gamma(\bar{t}-(3\delta/4))|, |\theta_\gamma(\bar{t}+(3\delta/4))|\}$. Thus there exists s'_γ such that $\theta_\gamma(s'_\gamma) = \theta_\gamma(s_\gamma)$. This implies that s'_γ is another switching time for γ. Indeed $\lambda_\gamma(t) \cdot G(\gamma(t)) = \lambda_\gamma(\bar{t}) \cdot v(G(\gamma(t)), t; \bar{t})$ and the vectors $v(G(\gamma(s_\gamma)), s_\gamma; \bar{t})$ and $v(G(\gamma(s'_\gamma)), s'_\gamma; \bar{t})$ are parallel, hence from $\lambda_\gamma(s_\gamma) \cdot G(\gamma(s_\gamma)) = 0$ we deduce $\lambda_\gamma(s'_\gamma) \cdot G(\gamma(s'_\gamma)) = 0$. Moreover, from $s'_\gamma \neq t_\gamma$ we conclude $\frac{d}{ds}\lambda_\gamma(s) \cdot G(\gamma(s))|_{s=s'_\gamma} \neq 0$. ■

Now we describe all possible Frame Points occurring in a strip. We start analyzing Frame Points on C and $\bar{C}$ curves. For doing this, let x be a point of a Frame Curve F of kind C or $\bar{C}$.

Proposition 14 *Consider a Frame Curve F of C or $\bar{C}$ type, a point $x \in F$ and denote by $\dot{F}$ the tangent unit vector to the curve F. If $\Delta_A(x) \cdot \Delta_B(x) \neq 0$, $X(x) \wedge \dot{F}(x) \neq 0$ and $Y(x) \wedge \dot{F}(x) \neq 0$ then x is not a Frame Point.*

Proof. For each point y in a neighborhood of x we indicate by (γ_y, λ_y) the extremal pair reaching y at time t_y. If $\theta_y(t) = arg(G(y), v(G(\gamma_y(t)), t; t_y)$ then $\mathrm{sign}(\dot{\theta}_y(t_y)) = \mathrm{sign}(\Delta_B(y))$. Hence, by continuity $\mathrm{sign}(\dot{\theta}_y(t)) \neq 0$ for (y, t) in a neighborhood of (x, t_x) and all trajectories γ_y switch. Finally, the nontangency assumptions for the vector fields X and Y ensure that Frame Curve F does not change type (from C to $\bar{C}$ or viceversa) and thus x is not a Frame Point. ■

Since abnormal extremals have been considered in the previous Section we assume $\Delta_A(x) \neq 0$. Hence if x is a Frame Point on the curve F then we have one of the following cases:

i) $\Delta_B(x) = 0$

ii) $X(x) \wedge \dot{F}(x) = 0$ or $Y(x) \wedge \dot{F}(x) = 0$,

and the generic assumption

(GA11) **i)** and **ii)** are mutually exclusive.

Case i) Assume i) happens and denote by T_B the unit tangent vector to the set $\Delta_B^{-1}(0)$. We choose a system of local coordinates so that $x = (0,0)$, $\dot{F} = (0,1)$ and $F \subset \{(x,y) : y \leq 0\}$. Let us denote by $I \in \{X, Y\}$ and $O \in \{X, Y\}$ the vector fields such that the extremal trajectories enter F as I-trajectories and exit F as O-trajectories.

If F is of type C then without loss of generality we assume that $I_1(x) \geq 0$ and $O_1(x) \geq 0$. We have two generic cases:

a) $I_2(x) > O_2(x)$,

b) $O_2(x) > I_2(x)$

and the generic assumption

(GA12) $I_2(x) \neq O_2(x)$

Let $K = \{aX(x) + bY(x) : a, b \in \mathbb{R}\}$ be the cone generated by $X(x)$ and $Y(x)$ (i.e. by $I(x)$ and $O(x)$). For $T_B(x)$ we have three possibilities: 1) $\pm T_B(x) \in K$; 2) $\pm T_B(x) \notin K$, $T_B(x)$ or $-T_B(x)$ is in the first orthant; 3) $\pm T_B(x) \notin K$, $T_B(x)$ or $-T_B(x)$ is in the second orthant, and the generic assumption

(GA13) $\pm T_B \notin \partial K, \quad \pm T_B \cdot e_1 \neq 0, \quad \pm T_B \cdot e_2 \neq 0.$

In case a)1) a turnpike generates from x and x is a $(S,C)_1$ Frame Point. In cases a)2)and 3) applying Theorem 8 we obtain that the extremal trajectories must switch before (case 2)) or after (case 3)) crossing F. Thus a second switching points curve generates at x and x is a $(C,C)_1$ point. Moreover, in both cases the I-trajectory trough x is a γ_0 Frame Curve.

The case b)1) is impossible (indeed the extremal trajectories would not switch). In cases b)2) and b)3) applying again Theorem 8 we obtain that the extremal trajectories must switch before (case 2)) or after (case 3)) crossing F. Thus a second switching points curve generates at x. This new Frame Curve can be of type C or $\bar{C}$. In case b)2) with a new C curve generated the point x is of type $(C,C)_1$ and the I-trajectory trough x is a γ_0 Frame Curve. While if the new curve is a $\bar{C}$ Frame Curve then we say that x is a $(C,\bar{C})_2$ point, the I-trajectory trough x is a γ_0 Frame Curve before x and a W Frame Curve after x, and x is a fold point. The case b)3) with a new curve of type C is entirely similar to case b)2) with a new C curve. While if the new curve is of type $\bar{C}$ then we say that the point x is of type $(C,\bar{C})_1$, the I-trajectory trough x is a γ_0 Frame Curve before x and a W Frame Curve after x, and x is a cusp point. All possibilities are depicted in figures 2.10 and 2.11.

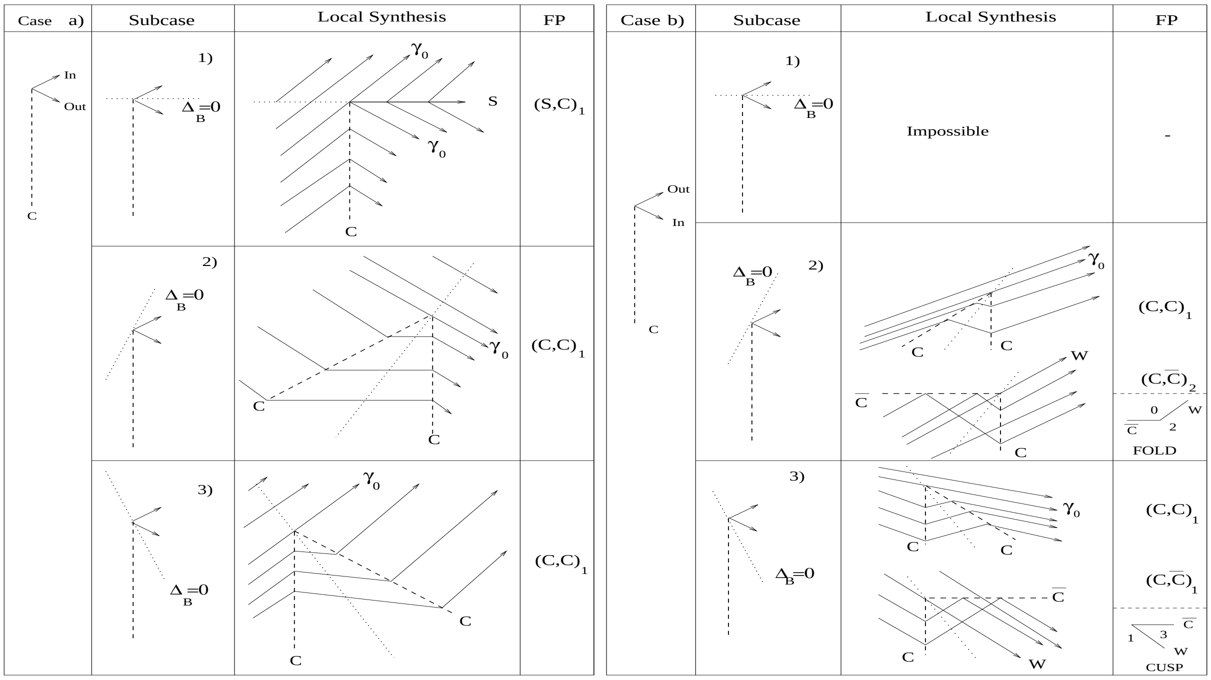


Figure 2.10 and 2.11: FPs along C

The case in which F is a $\bar{C}$ Frame Curve can be treated similarly. We have four possible cases and Frame Points of type $(C,\bar{C})_2$, $(\bar{C},S)_2$, $(C,\bar{C})_1$ and $(\bar{C},S)_1$ respectively. All cases are shown in figures 2.12 and 2.13.

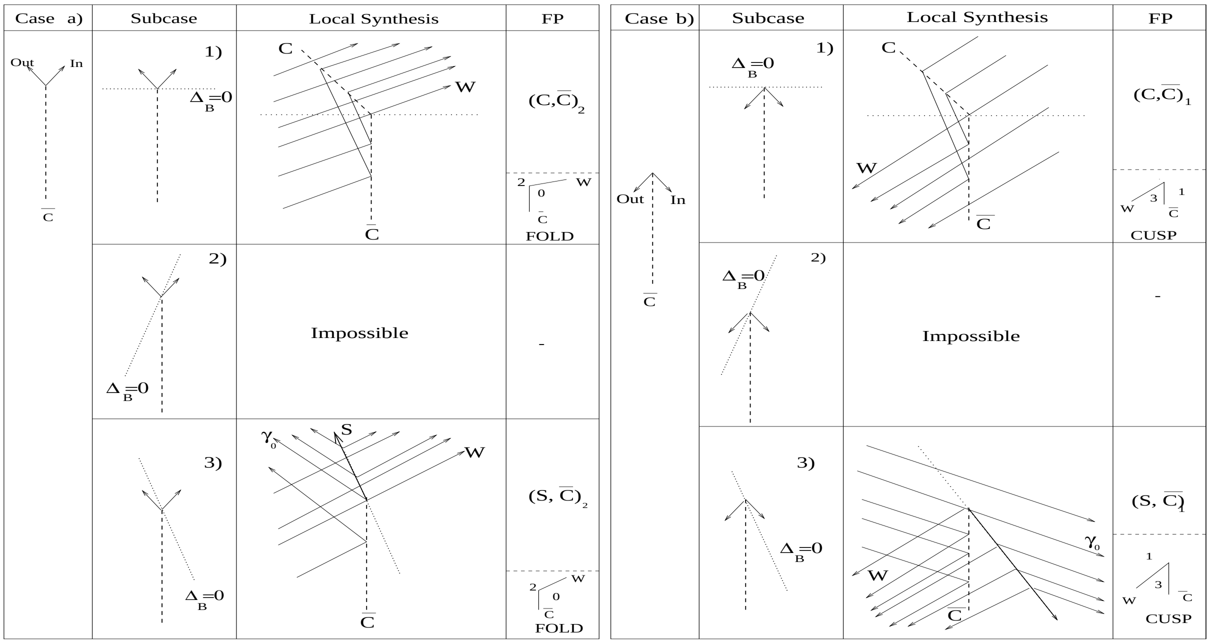


Figure 2.12 and 2.13: FPs along $\bar{C}$

Case ii) We have two possible subcases. Necessarily the vector $O(x)$ is tangent to F at x. We have the generic condition:

(GA14) there exists a neighborhood U of x such that the forward O-trajectory from x is contained either in the connected component of $U \setminus F$ where $I(x)$ points into or in the other connected component of $U \setminus F$.

In the former case of **(GA14)** we say that x is a $(C, \bar{C})_3$ Frame Point, the extremal trajectory through x is a γ_0 Frame Curve before x and a W Frame Curve after, and x is a fold point. In the latter case we call x a $(C, \bar{C})_4$ Frame Point, again the extremal trajectory through x is a γ_0 Frame Curve before x and a W Frame Curve after, while x is a cusp point. In figure 2.14 are listed the possibilities.

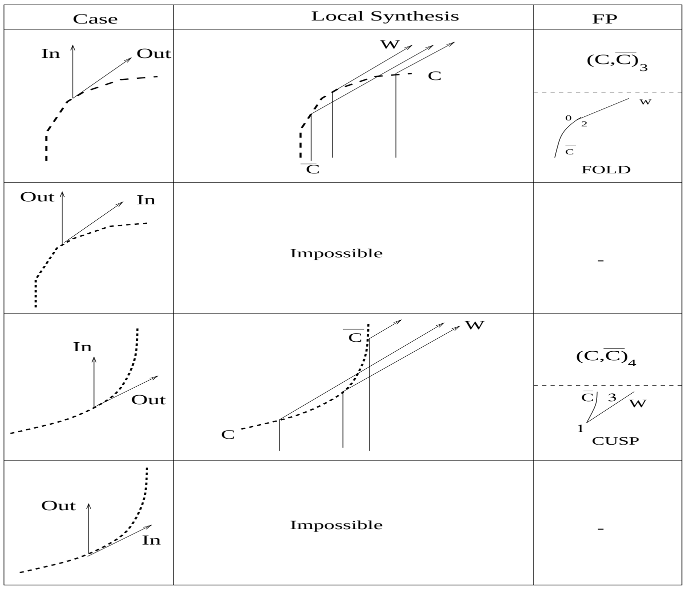


Figure 2.14: FPs along C and $\bar{C}$ (tangent cases)

2.4.2 Frame Curves and Points along γ_0

A Frame Curve F of type γ_0 that is a strip border generically does not enter a turnpike (even it may cross the set $\Delta_B^{-1}(0)$). Then if the trajectory γ_0 does not switch then there is no Frame Point. Assume then that x is a switching point for γ_0 then two switching curves C^1 and C^2 forms on the two sides of F. The curves intersect at x but generically we have:

(GA15) $\dot{C}^1(x)$ and $\dot{C}^2(x)$ are not parallel at x.

Let us denote by $I \in \{X, Y\}$ and $O \in \{X, Y\}$ the vector fields such that the extremal trajectory corresponding to F reaches x as I-trajectory and leaves from x as O-trajectories.

We choose a local system of coordinates so that $x = (0,0)$, $C^1 = \{(x,y) : y = 0, x \leq 0\}$, $I(x) = (0,1)$ and $C^2 \subset \{(x,y) : x \geq 0\}$. We distinguish two cases:

i) $\dot{C}^2$ or $-\dot{C}^2$ is in the second orthant,

ii) $\dot{C}^2$ or $-\dot{C}^2$ is in the first orthant.

Then there exists a small neighborhood U of x such that if r_1 and r_2 are the lines through the origin with tangent vectors $\dot{C}_1(x)$ and $\dot{C}_2(x)$ respectively, then $U \setminus (r_1 \cup r_2)$ has four connected components $U_1, \dots, U_4$. There are four generic subcases for $O(x)$ lying in one of the regions $U_1, \dots, U_4$. All the possibilities are shown in figure 2.15.

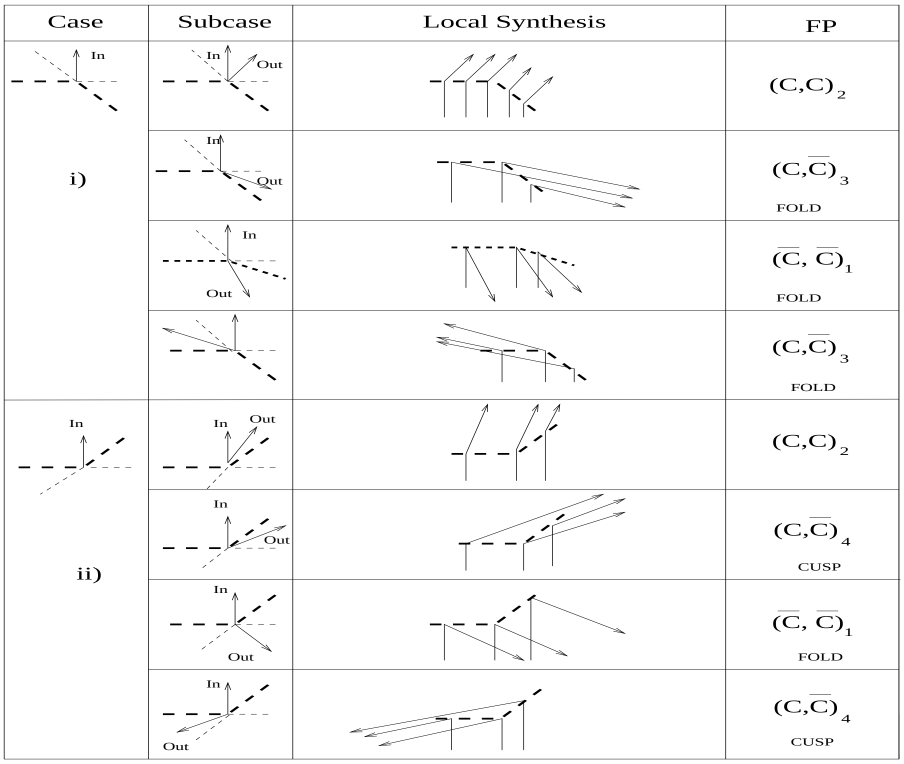


Figure 2.15: FPs along γ_0

2.4.3 Frame Curves and Points along W

Let us now consider the Frame Points along a Frame Curve F of type W. Also in this case we have a Frame Point only when the trajectory corresponding to W switches. Then assume that x is such a switching point. There exists two switching curves C_1 and C_2 generating at x with generically distinct tangent vectors $\dot{C}^1(x)$ and $\dot{C}^2(x)$ at x. In this case the two curves lie on the same side of W. Again, let us denote by $I \in \{X,Y\}$ and $O \in \{X,Y\}$ the vector fields such that the extremal trajectory corresponding to F

reaches x as I-trajectory and leaves from x as O-trajectories. We choose a local system of coordinates so that $x = (0,0)$, $I(x) = (1,0)$ and the extremal trajectories near F are contained in the set $\{(x,y) : y \geq 0\}$. We have two generic cases:

i) $O_2(x) > 0$,

ii) $O_2(x) < 0$,

and the generic condition:

(GA16) $O_2(x) \neq 0$.

There exists a small neighborhood U of x such that if r is the line through the origin parallel to $O(x)$ then the set $(U \cap \{(x,y) : y \geq 0\}) \setminus r$ in case i) and $(U \cap \{(x,y) : y \leq 0\}) \setminus r$ in case ii) has two connected components U_1 and U_2. Therefore, we have three subcases according to the fact that the vectors tangent to the curves C_1 and C_2 (or their opposite) both point into U_1, both point into U_2 or point to different connected components. All possible Frame Points are illustrated in figures 2.16 and 2.17

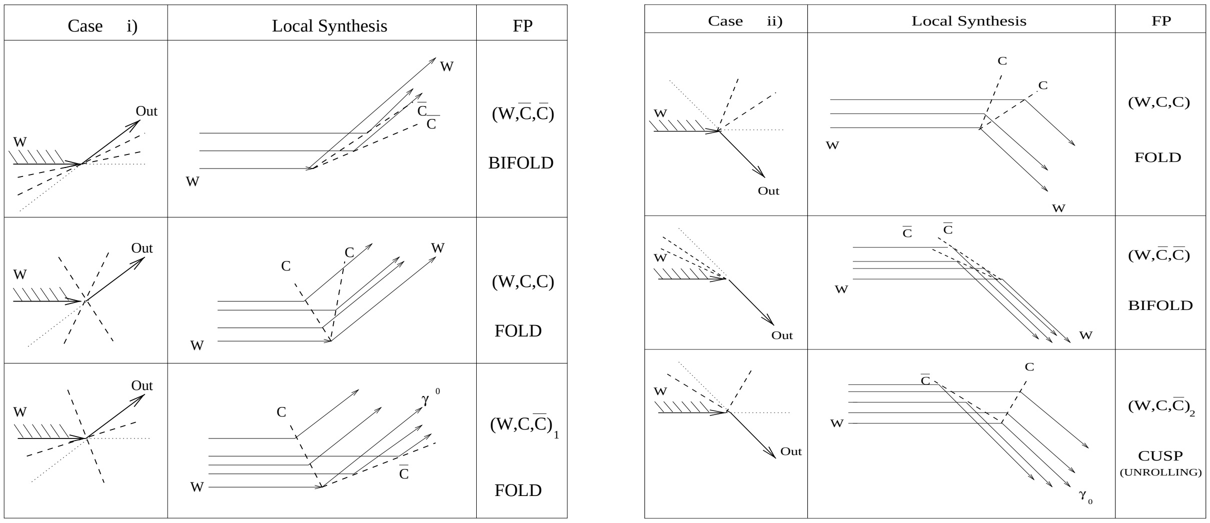


Figure 2.16 and 2.17: FPs along W (first and second case)

2.5 Extremal Synthesis

In this Section we prove that the algorithm of Section 2.2 gives rise to an extremal synthesis. First we describe the extremal locus near the origin and $\gamma^{\pm}$. Then we consider the possible intersections of lift of strips in the cotangent bundle. Finally we prove Theorem 1.

Observe that for every point $x \in Supp(\gamma^+) \cup Supp(\gamma^-)$ there are more than one p's such that $(x,p) \in N$. where π is the canonical projection from $\mathbb{R}^2 \times S^1$ to $\mathbb{R}^2$. For every $t > 0$ we have $\{(x,p) \in N : x = \gamma^+(t)\} = [\theta_1(t), \theta_2(t)]$, where the segment is taken in counterclockwise direction on S^1. This is explained in Chapter 4 in which it is explicitly computed a surface N_0 such that:

- $N|_{\pi_N^{-1}(I)}$ is homeomorphic to $N_0|_{\pi_{N_0}^{-1}(I)}$,
- $\sharp\pi_N^{-1}(x) = \sharp\pi_{N_0}^{-1}(x)$, $x \in I$,

where I is a neighborhood of the origin in $\mathbb{R}^2$ and π_N, π_{N_0} are the canonical projections of N and N_0 to $\mathbb{R}^2$ (cfr. Section 4.1 and Observation 4).

We define two curves in $\mathbb{R}^2 \times S^1$ both projecting on γ^+ by $\gamma^{1+}(t) := (\gamma^+(t), \theta_1(t))$, $\gamma^{2+}(t) := (\gamma^+(t), \theta_2(t))$. We do the same for γ^-. The FP (X,Y) on the plane (see Section 2.1.2) corresponds to a couple of FPs in the cotangent bundle called $(\gamma^{1+}, \gamma^{2-})$, $(\gamma^{2+}, \gamma^{1-})$. This can be easily understood from figure 2.18 that shows N in a neighborhood of the origin and a subset of $\mathbb{R}^2$ to which it is diffeomorphic. The others FPs on the curve γ^+ on the plane corresponds to some FPs of N on the curves γ^{1+} or γ^{1-} according to the direction toward which the corresponding FC bifurcates from γ^+. More details are give in Chapter 4.

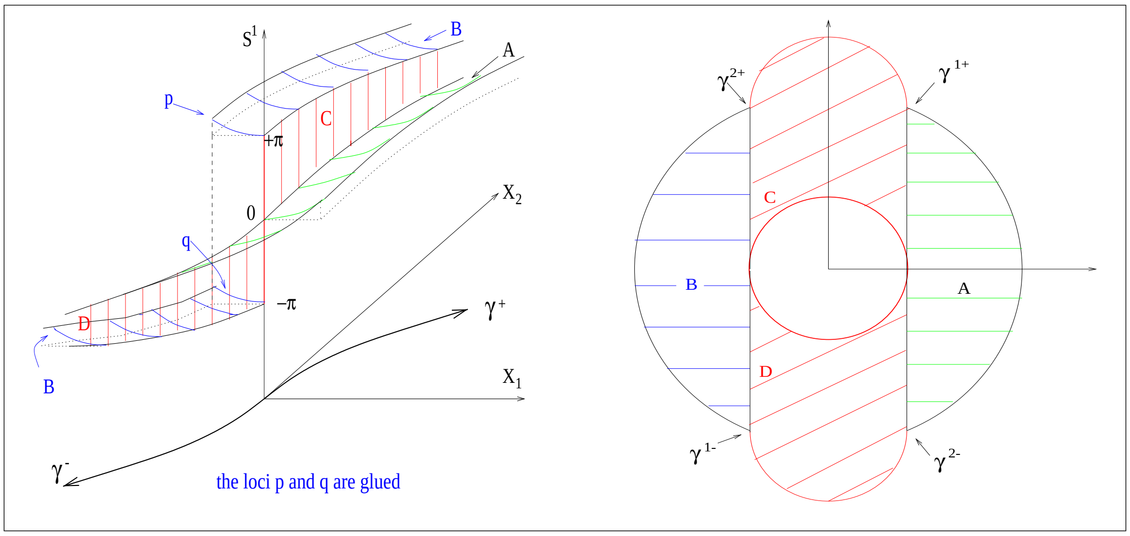


Figure 2.18: The Origin

Definition 18 *Let $\mathcal{Q}$ be the set of points $(x,p) \in N$ (we recall that $N = \pi_*(\mathcal{N})$) such that there exist two times $t_1, t_2 \leq \tau$, $d \in [0, min(t_1,t_2)[$, $\varepsilon > 0$ and two extremal pairs (γ_1, λ_1), (γ_2, λ_2) such that:*

$$\textbf{(1)} \quad \gamma_1(0) = \gamma_2(0) = 0, \;\; \gamma_1(t_1) = \gamma_2(t_2) = x, \;\; \lambda_1(t_1) = \lambda_2(t_2) = p \in S^1 \subset \mathbb{R}^2;$$

(2) γ_1 *and* γ_2 *correspond to the same control in resp.* $[t_1 - d, t_1]$, *and* $[t_2 - d, t_2]$;

(3) γ_1 *corresponds to the singular control* φ *in* $[t_1 - d - \varepsilon, t_1 - d]$; γ_2 *corresponds to the constant control +1 or to the constant control -1 in* $[t_1 - d - \varepsilon, t_1 - d]\}$;

The projection of $\mathcal{Q}$ on $\mathcal{R}(\tau)$ is showed in figure 2.19. The solid lines and the dashed lines describe two different strips.

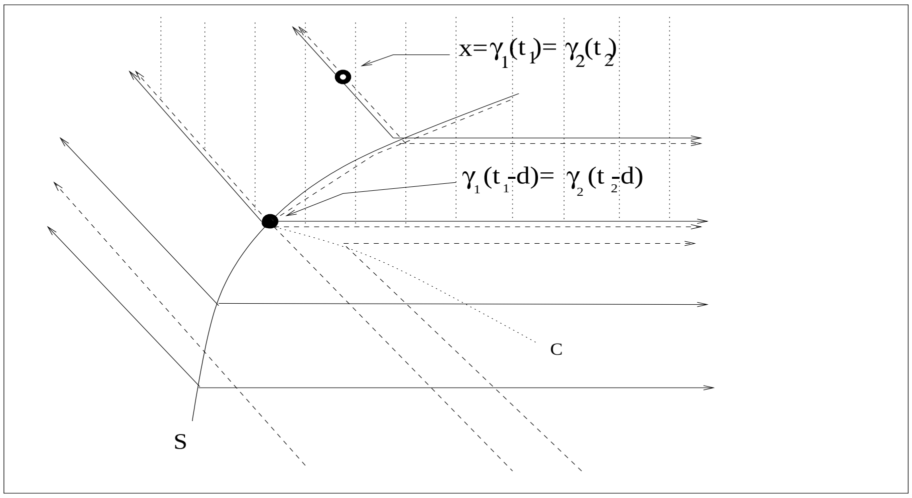


Figure 2.19: The projection of $\mathcal{Q}$ is the dotted region

Theorem 9 *Let* (x, p) *belong to* N*, then generically there exists a unique extremal pair* $(\gamma, \lambda) : [0, t] \to \mathbb{R}^4$ *such that* $\gamma(0) = 0$*,* $\gamma(t) = x$*,* $\lambda(t) = p \in S^1$ *for some* $t \leq \tau$ *iff* $(x, p) \notin \mathcal{Q}$.

Proof. By definition, the uniqueness implies $(x, p) \notin \mathcal{Q}$.
Let us now prove that generically $(x, \lambda) \notin \mathcal{Q}$ implies the uniqueness of (γ, λ). Assume by contradiction that $(x, p) \notin \mathcal{Q}$ and suppose that there exist two different extremal pairs (γ_1, λ_1), (γ_2, λ_2) such that $\gamma_1(0) = \gamma_2(0) = 0$, $\gamma_1(t_1) = \gamma_2(t_2) = x$, $\lambda_1(t_1) = \lambda_2(t_2) = p$ for some $t_1,\ t_2 \leq \tau$. Let $A := \{t \geq 0 :\ \ \gamma_1(t_1 - t) = \gamma_2(t_2 - t),\ \lambda_1(t_1 - t) = \lambda_2(t_2 - t)\}$ and A_c the connected component of A containing 0. Notice that A_c is closed and let $d := \max A_c$. If $d = \min(t_1, t_2)$ (that implies $t_1 = t_2$) then $(\gamma_1, \lambda_1) \equiv (\gamma_2, \lambda_2)$ and there is nothing to prove. Otherwise set:

$$
\begin{aligned}
\bar{t}_1 &= t_1 - d, \quad \bar{t}_2 = t_2 - d, \\
\bar{x} &= \gamma_1(\bar{t}_1) = \gamma_2(\bar{t}_2), \\
\bar{\lambda} &= \lambda_1(\bar{t}_1) = \lambda_2(\bar{t}_2).
\end{aligned}
$$

Claim: $\bar{\lambda} \cdot G(\bar{x}) = 0$

Proof of the Claim: let u_1 and u_2 be the controls corresponding to γ_1 and γ_2. There exists $\varepsilon > 0$ such that one of the following cases occurs:

(a) $u_1 = \varphi$ on $[\bar{t}_1 - \varepsilon, \bar{t}_1]$, $u_2 = +1$ or $u_2 = -1$ on $[\bar{t}_2 - \varepsilon, \bar{t}_2]$ or viceversa;

(b) $u_1 = +1$ on $[\bar{t}_1 - \varepsilon, \bar{t}_1]$, $u_2 = -1$ on $[\bar{t}_2 - \varepsilon, \bar{t}_2]$, or viceversa.

In case **(a)** from $u_1 = \varphi$ in $[\bar{t}_1 - \varepsilon, \bar{t}_1]$ we have $\bar{\lambda} \cdot G(\bar{x}) = 0$. In case **(b)** (say with $u_1 = +1$) from the PMP we have:

$$\begin{aligned}
\lambda_1(t) \cdot (F+G)(\gamma_1(t)) &= \max_{|w|\leq 1} \lambda_1(t) \cdot (F+wG)(\gamma_1(t)) \text{ for a.e. } t \in [\bar{t}_1 - \varepsilon, \bar{t}_1], \\
\lambda_2(t) \cdot (F-G)(\gamma_2(t)) &= \max_{|w|\leq 1} \lambda_2(t) \cdot (F+wG)(\gamma_2(t)) \text{ for a.e. } t \in [\bar{t}_2 - \varepsilon, \bar{t}_2],
\end{aligned}$$

so we have:

$$\begin{aligned}
\lambda_1(t) \cdot G(\gamma_1(t)) \geq 0 \text{ for a.e. } t \in [\bar{t}_1 - \varepsilon, \bar{t}_1], \\
\lambda_2(t) \cdot G(\gamma_2(t)) \leq 0 \text{ for a.e. } t \in [\bar{t}_2 - \varepsilon, \bar{t}_2].
\end{aligned}$$

Being all the functions continuous we get $\bar{\lambda} \cdot G(\bar{x}) = 0$ that proves the Claim.

Now since $(x,p) \notin \mathcal{Q}$ then, condition (3) of Definition 18 must fail. This means that we are in situation **(b)** of the proof of the Claim. Hence $(\gamma_1, \lambda_1)|_{[0,\bar{t}_1]}$ and $(\gamma_2, \lambda_2)|_{[0,\bar{t}_2]}$ are two different extremal pairs reaching $\bar{x}$, say with controls +1 and -1 respectively, and satisfying:

$$\lambda_1(\bar{t}_1) \cdot G(\gamma_1(\bar{t}_1)) = 0, \tag{2.4}$$

$$\lambda_2(\bar{t}_2) \cdot G(\gamma_2(\bar{t}_2)) = 0. \tag{2.5}$$

Generically $\Delta_A(x) \neq 0$ otherwise γ_1 and γ_2 would be two NTAE, see Section 2.3. We have the following cases:

Case 1 $\bar{x} \notin \Delta_B^{-1}(0)$. In this case from Proposition 5, $\bar{t}_1$ is a switching from Y to X for γ_1 and $\bar{t}_2$ is a switching from X to Y for γ_2 (hence generically $d = 0$). Define $v^i(t) := v^{\gamma_i}(G(\gamma_i(t)), t; \bar{t}_i)$ $(i = 1,2)$ (cfr. subsection 1.1) and the functions:

$$\begin{aligned}
\alpha^1(t) &:= arg(G(\bar{x}), v^1(t)) \\
\alpha^2(t) &:= arg(G(\bar{x}), v^2(t)).
\end{aligned}$$

Now $\alpha^i(t) = \theta^{\gamma_i}(t) + A^i$ $(i = 1,2)$ where A^1, A^2 are real numbers. Since $\Delta_B(\bar{x}) \neq 0$, by Proposition 5 there exists $\varepsilon > 0$ such that $sgn(\dot{\alpha}^1(\bar{t}_1 + h)) = sgn(\dot{\alpha}^2(\bar{t}_2 + h))$ for each $h \in [-\varepsilon, \varepsilon]$. Moreover we can choose ε small enough such that:

C1) for each $t \in [\bar{t}_1 - \varepsilon, \bar{t}_1]$, $\lambda_1(t) \cdot G(\gamma_1(t)) = \lambda_1(\bar{t}_1) \cdot v^1(t) \geq 0$ where the equality holds iff $t = \bar{t}_1$;

C2) for each $t \in [\bar{t}_2 - \varepsilon, \bar{t}_2]$, $\lambda_2(t) \cdot G(\gamma_2(t)) = \lambda_2(\bar{t}_2) \cdot v^2(t) \leq 0$ where the equality holds iff $t = \bar{t}_2$.

Now $\lambda_1(\bar{t}_1) = \lambda_2(\bar{t}_2)$ and v^1, v^2 are two vector functions satisfying $v^1(\bar{t}_1) = v^2(\bar{t}_2)$ and rotating in the same direction. This contradicts **C1)** or **C2)**.

Case 2 $\bar{x} \in \Delta_B^{-1}(0)$ and X, Y point to the same side of $\Delta_B^{-1}(0)$. In this case with an entirely similar argument used for Case 1 we reach a contradiction.

Case 3 $\bar{x} \in \Delta_B^{-1}(0)$ and X, Y point to opposite side of $\Delta_B^{-1}(0)$. In this case from Proposition 3 $\Delta_B^{-1}(0)$ is a turnpike near x, hence γ_1 and γ_2 may enter the turnpike at time respectively $\bar{t}_1$ and $\bar{t}_2$. This case is not generic. ∎

Let us now introduce the concept of equivalent extremal syntheses in finite time and stable extremal synthesis. An extremal synthesis at time $\tau > 0$ is the intersection of an extremal synthesis (see Definition 7) with the set $\{(x, p) : x \in R(\tau)\}$.

Definition 19 *Let Γ_1^* and Γ_2^* be two extremal syntheses in finite time corresponding to $(F_1, G_1), (F_2, G_2) \in \Xi$ respectively. We say that Γ_1^* and Γ_2^* are equivalent and we write $\Gamma_1^* \sim \Gamma_2^*$ if there exists an homeomorphism $\Psi : \mathbb{R}^4 \to \mathbb{R}^4$ such that:*

- $\Psi(\mathcal{N}_1) = \Psi(\mathcal{N}_2)$;
- *Ψ maps arcs of extremal trajectories corresponding to control ± 1 (resp. φ) to arcs of extremal trajectories corresponding to ± 1 (resp. φ);*
- *$\pi_* \circ \Psi \circ \pi_*^{-1}$ maps FCs (resp. FPs) of N_1 to FCs (resp. FPs) of N_2 of the same kind.*

Definition 20 *An extremal synthesis Γ^* corresponding to $(F, G) \in \Xi$ is stable at time τ if the following holds. There exists ε such that if $\|(F', G') - (F, G)\| < \varepsilon$, $|\tau' - \tau| < \varepsilon$ and ${\Gamma^*}'_{\tau'}$ is an extremal synthesis at time τ' for (F', G') then $\Gamma^*_\tau \sim {\Gamma^*}'_{\tau'}$ (where Γ^*_τ is the the extremal synthesis at time τ).*

Proof of Theorem 2. We define Γ^* to be the set of extremal pairs constructed by the Algorithm. First we fix $\tau \in]0, +\infty[$ satisfying **(GAτ)**. The Algorithm of Section 2.2 constructs all the extremal pairs up to time τ. This guarantees 1. of Definition 7 at time τ. Under the generic conditions **(P1)**–**(P9)**, **(GA1)**–**(GA16)**, **(GAτ)**, **(GAφ)** the Algorithm stops after a finite number of steps generating a finite number of strips, FCs and FPs. This ensures 2. of Definition 7. Moreover there exists $m \in \mathbb{N}$ such that for every trajectory γ, built by the Algorithm, we have $n(\gamma) \leq m$, that is γ is a concatenation of at most m arcs and each arc corresponds to a constant control ± 1 or to the singular control φ. Hence we obtain 3. of Definition 7. The lift to $\mathbb{R}^2 \times S^1$ of each strip $\mathcal{S}$ is a piecewise–$\mathcal{C}^1$ manifold and there is a natural stratification setting FCs be the one dimensional strata and FPs the zero dimensional strata. Now for each couple of (lift of) strips we define

$$\mathcal{S}^1 \cap \mathcal{S}^2 = \Big\{(x,p) = (\gamma_1, \lambda_1)(t_1) = (\gamma_2, \lambda_2)(t_2) : \ (\gamma_i, \lambda_i) \in \mathcal{S}^i, \ t_i \in Dom(\gamma_i), \ i \in \{1,2\}\Big\},$$

and

$$\mathcal{S}^1 \cup \mathcal{S}^2 = \Big\{(x,p) = (\gamma_i, \lambda_i)(t_i) : \ (\gamma_i, \lambda_i) \in \mathcal{S}^i, \ t_i \in Dom(\gamma_i), \ i \in \{1,2\}\Big\}.$$

From Theorem 9 we get a description of the set $\mathcal{S}^1 \cap \mathcal{S}^2$, more precisely if two strips intersect each other then $\mathcal{S}^1 \cap \mathcal{S}^2 = \mathcal{Q}(\mathcal{S}^1, \mathcal{S}^2)$ where $\mathcal{Q}(\mathcal{S}^1, \mathcal{S}^2) = \mathcal{Q} \cap (\mathcal{S}^1 \cup \mathcal{S}^2)$. It follows that $\mathcal{S}^1 \cup \mathcal{S}^2$ is a stratified set with FCs and FPs as strata, indeed $\partial \mathcal{Q}(\mathcal{S}^1, \mathcal{S}^2) \subset FCs$. Since the number of strips is finite we conclude that N is a stratified set in time τ. Now $\mathcal{N} = \pi_*^{-1}(N)$ therefore $\mathcal{N}$ is a locally closed locally finite in $\mathbb{R}^4$ regular Whitney stratified subset of $\mathbb{R}^4$ of dimension three. Thus 4. of Definition 7 is verified. The set of discontinuities of $\sharp(\Gamma^*(x,p))$ on $\mathcal{N}$ is precisely the union of the sets $\pi_*^{-1}(\partial \mathcal{Q}(\mathcal{S}^1, \mathcal{S}^2))$. Hence we obtain 5. of Definition 7. Since τ is arbitrary (in the set of time satisfying **(GAτ)**) we conclude that Γ^* is an extremal synthesis.

The generic conditions **(P1)**–**(P9)**, **(GA1)**–**(GA16)**, **(GAφ)**, **($\mathcal{A}$)**, and **($\mathcal{F}$)**, imply that all FCs and FPs are stable at a.e. time τ. Therefore for a.e. fixed time τ, Γ^* is stable and this concludes the proof. ■

2.6 Classification of the singularities

2.6.1 Classification of the Singularities in $\mathbb{R}^2 \times S^1$

In this Section we topologically classify all generic FCs and FPs in the normalized cotangent bundle $\mathbb{R}^2 \times S^1$.

Definition 21 *Let l_1 and l_2 be two FCs of $N = \pi_*(\mathcal{N})$. We say that l_1 and l_2 are topologically equivalent if for every $(x_1, p_1) \in l_1$ and for every $(x_2, p_2) \in l_2$ there exist a neighborhood U_1 of (x_1, p_1) in N, a neighborhood U_2 of (x_2, p_2) in N and a homeomorphism $\Psi : U_1 \to U_2$ such that:*

1) $\Psi(U_1) = U_2$;
2) $\Psi(l_1 \cap U_1) = l_2 \cap U_2$;
3) $\Psi(x_1, p_1) = (x_2, p_2)$;
4) Ψ *send extremal arcs of trajectory–covector pairs onto extremal arcs of trajectory–covector pairs.*

Definition 22 *Let (x,p_1), (x_2,p_2) be two FPs of N. We say that they are topological equivalent if there exist a neighborhood U_1 of (x_1,p_1) in N, a neighborhood U_2 of (x_2,p_2) in N and a homeomorphism $\Psi : U_1 \to U_2$ satisfying $1,3,4$ of Definition 21 and:*

5) if l_1 is a FC that intersects U_1 then $\Psi(l_1 \cap U_1) = l_2 \cap U_2$ where l_2 is a FC equivalent to l_1.

By the previous analysis and from Definition 21 we have the following equivalence:

- $\gamma^{1\pm}|_{[s_i^\pm,t_i^\pm]} \sim \gamma^{2\pm}|_{[s_i'^\pm,t_i'^\pm]}$. We call this class TY (topologically equivalent to Y–curve). See also Figure 2.18 and Chapter 4;

- $\gamma^{1\pm}|_{[t_i^\pm,s_{i+1}^\pm]} \sim \gamma^{2\pm}|_{[t_i'^\pm,s_{i+1}'^\pm]} \sim \gamma^0 \sim W \sim W^C \sim W^D$ (for every γ^0, W, W^C, W^D that are not in $\partial\mathcal{Q}(\mathcal{S}^1,\mathcal{S}^2)$ for every $\mathcal{S}^1,\mathcal{S}^2$ extremal strips). We call this class $T\gamma^0$;

- all C and $\bar{C}$ are equivalent. We call this class TC;

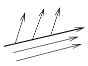
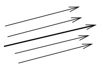
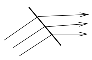
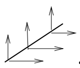
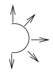
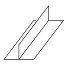
- all the turnpikes are equivalent. We call this class TS;

- the two connected components of $\pi^{-1}(0) \setminus \{(\gamma^{1+},\gamma^{2-}),(\gamma^{2+},\gamma^{1-})\}$ are equivalent. We call this class TB;

- all the $\partial\mathcal{Q}(\mathcal{S}^1,\mathcal{S}^2)$ for every $\mathcal{S}^1,\mathcal{S}^2$ extremal strips, are equivalent. We call this class $T\partial\mathcal{Q}$.

It follows:

Theorem 10 *The equivalence classes of FCs in $\mathbb{R}^2 \times S^1$ are the following:*

TY , Tγ⁰ , TC , TS . TB , T∂Q ,

Similarly from Definition 22 we have the equivalences:

- $(\gamma^{1+},\gamma^{2-}) \sim (\gamma^{2+},\gamma^{1-})$. We call this class $T(X,Y)$.

- All the FPs of kind $(Y,C)_2$, $(Y,C)_2^{tg}$, $(Y,\bar{C})_1$, $(Y,\bar{C})_1^{tg}$, $(Y,\bar{C})_1^{t-0}$ are equivalent. We call this class $T(Y,C)_2$.

- All the FPs of kind $(Y,C)_3$, $(Y,\bar{C})_2$, $(Y,\bar{C})_3^{tg}$ are equivalent. We call this class $T(Y,C)_3$.

- All the FPs of kind $(C,C)_1$, $(C,\bar{C})_2$, $(C,\bar{C})_1$ are equivalent. We call this class $T(C,C)_1$.

- All the FPs of kind $(C,C)_2$, $(\bar{C},\bar{C})_1$, $(C,\bar{C})_3$, $(C,\bar{C})_4$, $1_{ab}, \ldots, 24_{ab}$, (W,C,C), $(W,\bar{C},\bar{C})$, $(W,C,\bar{C})_1$, $(W,C,\bar{C})_2$ (where $1_{ab}...24_{ab}$ are the points of figure 2.6, 2.7, 2.8, 2.9) are equivalent. We call this class $T(C,C)_2$.

- All the FPs of kind $(S,C)_1$, $(S,\bar{C})_1$, $(S,\bar{C})_2$ are equivalent. We call this class $T(S,C)$.

- All the FPs of kind $(Y,C)_1$ are equivalent. We call this class $T(Y,C)_1$.

- All the FPs of kind (Y,S) are equivalent. We call this class $T(Y,S)$.

It follows:

Theorem 11 *The equivalence classes of FPs in $\mathbb{R}^2 \times S^1$ are the following: $T(X,Y)$, $T(Y,C)_1$, $T(Y,C)_2$, $T(Y,C)_3$, $T(C,C)_1$, $T(C,C)_2$, $T(S,C)$, $T(Y,S)$.*

2.6.2 Projection singularities

Consider the following

Definition 23 *Let (x_1,p_1), (x_2,p_2) be two points of N. We say that they are two equivalent topological singularities of the projection of the extremal synthesis, and we we write $(x_1,p_1) \sim_\pi (x_2,p_2)$ if there exists an homeomorphism $\Psi : U_1 \to U_2$, U_i neighborhood of (x_i,p_i) $(i=1,2)$ in $\mathbb{R}^2 \times S^1$, such that for every $p,p' \in S^1$ we have:*

$$\begin{array}{ll} 1) & \pi\left(\Psi(x,p)\right) = \pi\left(\Psi(x,p')\right) \\ 2) & \Psi(U_1 \cap N) = U_2 \cap N \end{array}$$

Recalling Definition 9, take $(x_1,p_1),(x_2,p_2)$ in N such that $x_1,x_2 \notin Supp(\gamma^+) \cup Supp(\gamma^-)$, and let γ_1,γ_2 be the two extremal trajectories such that γ_i reaches x_i with covector p_i. If the two points x_1,x_2 of $\mathcal{R}(\tau)$ are both normal, fold, cusp, bifold or ribbon, then it is easy to check that $(x_1,p_1) \sim_\pi (x_2,p_2)$. We call such equivalence classes respectively *normal, fold, cusp, bifold and ribbon*. See the figure 2.20.

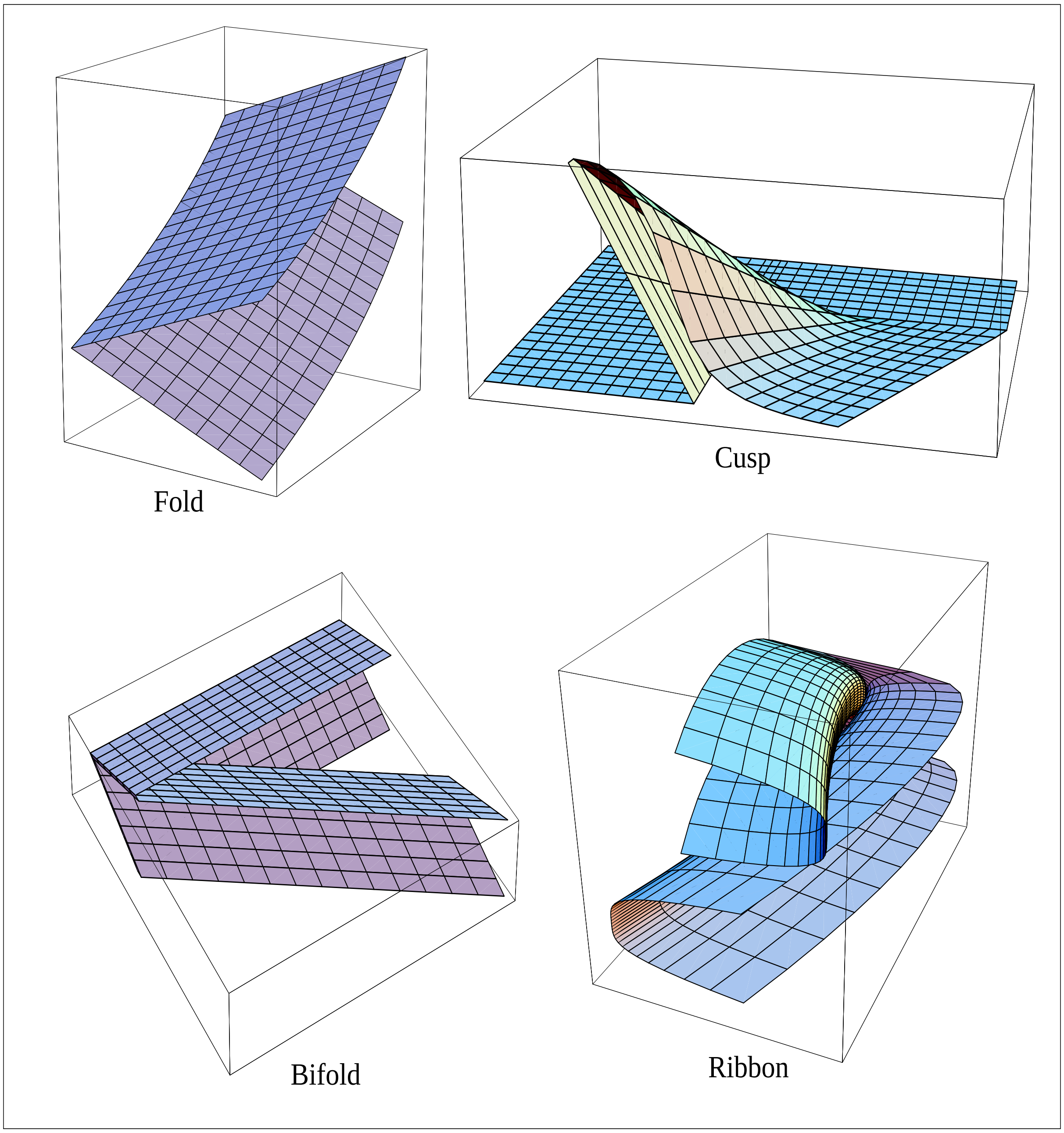


Figure 2.20: Projection Singularities

Notice that the difference, with respect to the Whitney classical singularities of smooth maps between two dimensional smooth manifolds, (see [40]), is that now we have a s-tratified set with "edges" and "corners". From the analysis of the beginning of Section 2.5 we have the following. There are projection singularities along $\gamma^+ \cup \gamma^-$ that we call *vertical*. In particular all points of $\pi^{-1}(0)$ are called *vertical origin*. All the points of $\pi^{-1}(Supp(\gamma^+) \cup Supp(\gamma^-))$ that are not FPs are equivalent and they are called *vertical normal* singularities. The lift in N of a FP of kind $(Y, \bar{C})_1^{tg}$ is called a *vertical cusp point* (see figure 2.2 and 2.3). It is easy to check the equivalences $(Y, \bar{C})_1^{tg} \sim_\pi (Y, \bar{C})_1^{t-o} \sim_\pi$

$(Y,\bar{C})_1$. The lifts of the equivalent points $(Y,C)_3 \sim_\pi (Y,C)_3^{tg}$ are called *vertical ending*. Finally the lift of $(Y,\bar{C})_2$ is called a *vertical fold*. From the analysis of Section 2.2, 2.3, 2.4 we obtain Theorem 2.

Chapter 3

Abnormal Extremals

In this Chapter we study the properties of abnormal extremals for the minimum time stabilization to the origin of the control system:

$$\dot{x} = F(x) + uG(x), \qquad x \in \mathbb{R}^2, \qquad u \in [-1, 1], \qquad F(0) = 0. \tag{3.1}$$

Definition 24 *Let (γ, λ) be an extremal pair. If the corresponding Hamiltonian satisfies:*

$$\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0 \text{ for almost every } t \in Dom(\gamma)$$

we say that (γ, λ) is an abnormal extremal.

In Chapter 2 it is proved that the *ribbon* singularity may occur only along abnormal extremals. In Section 3.2 we prove that the singularity is in fact realized. Moreover we prove the following Theorem that describes all the sequences of singularities along abnormal extremals.

Theorem 12 *Under generic assumptions on $(F, G) \in \Xi$, the set of abnormal extremals for the control problem (3.1) can be classified through a set of words recognizable by an automaton.*

For self consistency of the chapter we recall some definition of Chapter 2.

3.1 Properties of Abnormal Extremals

In this section we state and prove the main results about the switching strategies of abnormal extremals. We recall the following (cfr. Definition 14):

Definition 25 *Let $\gamma : [0, \tau] \to \mathcal{R}(\tau)$ be (the first component of) an abnormal extremal for the control problem (3.1) such that it switches at least one time and let t_1 be its first switching time. We refer to the couples (γ, t_1) as* Non Trivial Abnormal Extremal *(in the following NTAE). By definition a NTAE is maximal if defined on $[0, \tau]$.*

Moreover we have:

Definition 26 *Let $\gamma : [0,\tau] \to \mathbb{R}^2$ be a NTAE, and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. We set $AA(i) = Supp\left(\gamma|_{[t_i,t_{i+1}]}\right)$ $(i = 1, ..., n(\gamma) - 1)$ and we call it an abnormal arc.*

Theorem 13 *Let $\gamma : [0,\tau] \to \mathbb{R}^2$ be an extremal trajectory for the control problem (3.1) such that it switches at least one time, $\lambda : [0,\tau] \to \mathbb{R}_2$ the corresponding covector and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Then it holds:*

A) *$\lambda(.)$ is unique (up to the multiplication by a positive constant);*

B) *under generic assumptions the following conditions are equivalent:*

- **(a)** *(γ, t_1) is a NTAE;*
- **(b)** *$\gamma(t_i) \in \Delta_A^{-1}(0)$ for some $i \in \{1, ..., n(\gamma) - 1\}$;*
- **(c)** *$\gamma(t_i) \in \Delta_A^{-1}(0)$ for each $i \in \{1, ..., n(\gamma) - 1\}$;*
- **(d)** *$\gamma(\bar{t}) \in \Delta_A^{-1}(0)$ $(\bar{t} \in Dom(\gamma))$ iff $\bar{t} = t_i$ for some $i \in \{1, ...n(\gamma) - 1\}$;*

C) *under generic assumptions,* **a)** *(or equivalently* **(b)** *or* **(c)** *or* **(d)***) implies that for every interval $[a, b] \subset [0, \tau]$ $(a < b)$, γ does not correspond to the singular control φ.*

Proof.
Proof of **A)**. The covector associated to an extremal trajectory is completely determined after the first switching (see Theorem 20 of Chapter 4). From $n(\gamma) \geq 2$ it follows that $\lambda(.)$ is unique up to a positive constant.

Proof that **(a)** implies **(c)**. Let (γ, λ) be an abnormal extremal such that $n(\gamma) \geq 2$. For each t_i $(i = 1, ..., n(\gamma) - 1)$ we have $\lambda(t_i) \cdot G(\gamma(t_i)) = 0$ and there exists a sequence $t'_m \nearrow t_i$ such that

$$\mathcal{H}(\gamma(t'_m), \lambda(t'_m), u(t'_m)) = \lambda(t'_m) \cdot (F + u(t'_m)G)(\gamma(t'_m)) = 0.$$

Hence $\lambda(t'_m) \cdot F(\gamma(t'_m)) \to 0$ and $\lambda(t_i) \cdot F(\gamma(t_i)) = 0$. Being $\lambda(t_i) \neq 0$ we can conclude $F(\gamma(t_i)) = \alpha_i G(\gamma(t_i))$ for some $\alpha_i \neq 0$. It follows **(c)**.

Proof of **C)**. With the above proof, since $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0) \cap \mathcal{R}(\tau)$ is a finite set, generically if $\gamma \in \Gamma$ then it does not switch on $\Delta_A^{-1}(0) \cap \Delta_B^{-1}(0)$.

Proof that **(b)** implies **(a)**. Assume now that $\Delta_A(\gamma(t_i)) = 0$ for some $i \in \{1, ...n(\gamma) - 1\}$.

We have $\lambda(t_i) \cdot G(\gamma(t_i)) = 0$ and from $\Delta_A(\gamma(t_i)) = 0$ we get $\lambda(t_i) \cdot F(\gamma(t_i)) = 0$. There exists a sequence $t'_m \nearrow t_i$ such that:

$$\mathcal{H}(\gamma(t'_m), \lambda(t'_m), u(t'_m)) = \lambda(t'_m) \cdot (F + u(t'_m)G)(\gamma(t'_m)) \to 0.$$

From the PMP we know that $\mathcal{H}$ is almost everywhere equal to a fixed constant in $Dom(\gamma)$ hence we can conclude $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = 0$ almost everywhere.

Proof that **(a)** implies **(d)**. Fix $\bar{t}$ such that $\gamma(\bar{t}) \in \Delta_A^{-1}(0)$. We have $F(\gamma(\bar{t})) = \beta G(\gamma(\bar{t}))$ (by generic assumption we may assume $\beta \neq 0, \pm 1$) and there exists a a sequence $t'_m \nearrow \bar{t}$ such that $|u(t'_m)| = 1$ and

$$\mathcal{H}(\gamma(t'_m), \lambda(t'_m), u(t'_m)) = \lambda(t'_m) \cdot (F + u(t'_m)G)(\gamma(t'_m)) = 0.$$

Hence $(1+u(t'_m)\beta)\lambda(t'_m)\cdot G(t'_m) \to 0$ and being $\lim_{m\to\infty} u(t'_m) = \pm 1$ we have $\lambda(\bar{t})\cdot G(\bar{t}) = 0$. Generically $\Delta_B(\gamma(\bar{t})) \neq 0$ thus $\dot{\theta}(\bar{t}) \neq 0$ and $\bar{t}$ is a switching time (see Proposition 5). S-ince **a)** implies **c)** we get that $\Delta_A(\gamma(t_i)) = 0$ for each i. Thus we are done.

The implications **(c)**$\Rightarrow$**(b)**, **(d)**$\Rightarrow$**(b)** are obvious. This concludes the proof. ■

We recall that if (γ, t_1), is a NTAE then $\gamma \in \partial\Gamma_A \subset \partial\Gamma$ (cfr. Corollary 1 and Definition 2.3). Moreover every $AA(i)$ can be only of kind γ^0, W^C or W^D (cfr. Definition 17 and Proposition 11).

Theorem 14 *Let γ be a NTAE and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Moreover set $t_0 = 0$. Let β_i be defined by $F(\gamma(t_i)) = \beta_i G(\gamma(t_i))$ $(i = 0, ..., n(\gamma) - 1)$ then, under the generic conditions $\beta_i \neq \pm 1$ $(i = 1, ...n(\gamma) - 1)$, and it holds for all $i = 0, .., n(\gamma) - 2$:*

$$\begin{cases} v(G(\gamma(t_{i+1})), t_{i+1}; 0) = (\beta_i + 1)/(\beta_{i+1} + 1)v(G(\gamma(t_i)), t_i; 0) \text{ if } \gamma \text{ correspond to} \\ \qquad\qquad\qquad\qquad \text{constant control +1 in } [t_i, t_{i+1}] \\ v(G(\gamma(t_{i+1})), t_{i+1}; 0) = (\beta_i - 1)/(\beta_{i+1} - 1)v(G(\gamma(t_i)), t_i; 0) \text{ if } \gamma \text{ correspond to} \\ \qquad\qquad\qquad\qquad \text{constant control -1 in } [t_i, t_{i+1}] \end{cases}$$

Proof. Fix i and suppose that γ corresponds to constant control $+1$ in $[t_i, t_{i+1}]$, being the other case similar. From $F(\gamma(t_{i+1})) = \beta_{i+1}G(\gamma(t_{i+1}))$, recalling Proposition 4 we have

$$F(\gamma(t_i)) + G(\gamma(t_i)) = v(F(\gamma(t_{i+1})) + G(\gamma(t_{i+1})), t_{i+1}; t_i) = (1 + \beta_{i+1})v(G(\gamma(t_{i+1})), t_{i+1}; t_i).$$

Now from $F(\gamma(t_i)) = \beta_i G(\gamma(t_i))$ (notice that that in case $i = 0$ we have $F(0) = 0$, hence $\beta_0 = 0$) and using again Proposition 4 we have

$$(1 + \beta_i)v(G(\gamma(t_i)), t_i; 0) = (1 + \beta_{i+1})v(G(\gamma(t_{i+1})), t_{i+1}; 0)$$

that concludes the proof. ■

Theorem 15 *Let $\gamma : [0, \tau] \to \mathbb{R}^2$ be an extremal trajectory for the control problem (3.1) such that it switches at least one time, θ^γ the corresponding function defined in (1.10) and $t_1 < t_2 < ... < t_{n(\gamma)-1} < t_n(\gamma) := \tau$ the sequence of switching times. . Then under generic assumptions the following conditions are equivalent:*

(a) *(γ, t_1) is a NTAE;*

(b) *$\theta^\gamma(t_i) \in \{0, \pm\pi\}$ for some $i \in \{1, ..., n(\gamma) - 1\}$;*

(c) *$\theta^\gamma(t_i) \in \{0, \pm\pi\}$ for each $i \in \{1, ..., n(\gamma) - 1\}$;*

(d) *$\theta^\gamma(\bar{t}) \in \{0, \pm\pi\}$, $(\bar{t} \in Dom(\gamma))$ iff $\bar{t} = t_i$ for some $i \in \{1, ...n(\gamma) - 1\}$.*

Proof.
Proof that **(a)** implies **(c)**. From Theorem 14, by induction it follows the conclusion because $\theta^\gamma(0) = 0$.

Proof that **(c)** implies **(a)**. From **c)** we have $\theta^\gamma(t_1) \in \{0, \pm\pi\}$, then for some $b \in \mathbb{R}$ (that by genericity we may assume different from 0 and 1) it holds $v^\gamma(G(\gamma(t_1)), t_1; 0) = bG(\gamma(0))$. Now if we suppose that γ corresponds to constant control $+1$ in the interval $[0, t_1]$ (being the opposite case similar), we have $bG(\gamma(0)) = b(F+G)(\gamma(0)) = v^\gamma(b(F+G)(\gamma(t_1), t_1; 0)$. From the injectivity of the map $v_0 \to v^\gamma(v_0, t_0; t_1)$ we may conclude $\gamma(t_1) \in \Delta_A^{-1}(0)$. Using Theorem 13 it follows **(a)**.

Proof that **(b)** implies **(c)** and viceversa. Clearly **(b)** follows from **(c)** let us prove the contrary. Let λ be the covector associated to γ. From **(b)** we have that $\lambda(0)$ is orthogonal to $G(0)$, hence γ switches iff $\theta^\gamma \in \{0, \pm\pi\}$. Thus **(c)** follows.

Proof that **(a)** implies **(d)**. It is a consequence of the generic condition $\theta^\gamma(\bar{t}) = 0 \Rightarrow \dot{\theta}^\gamma(\bar{t}) \neq 0$.

The implication **(d)**$\Rightarrow$**(c)** is obvious. This concludes the proof. ■

Theorem 16 *Let (γ, t_1) be a NTAE and $0 =: t_0 < t_1 < t_2 < ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. Suppose that for some $i \in \{0, 1, ..., n(\gamma) - 1\}$, $\Delta_A^{-1}(0)$ is inverse at the points $\gamma(t_i)$, $\gamma(t_{i+1})$. Then $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i)$.*

Proof. Set $(F + G)(\gamma(t_i)) = \alpha_i(F - G)(\gamma(t_i))$ and $F(\gamma(t_i)) = \beta_i G(\gamma(t_i))$. Under generic assumptions, α_i and β_i are well defined for each $i = 1, ...n(\gamma) - 1$ and it holds:

$$\alpha_0 = -1, \quad \beta_0 = 0 \text{ and } \alpha_i, \beta_i \notin \{0, \pm 1\}, \quad \beta_i = \frac{1 + \alpha_i}{1 - \alpha_i}.$$

Now if $\Delta_A^{-1}(0)$ is inverse at both points $\gamma(t_i), \gamma(t_{i+1})$ $(i = 0, ..., n(\gamma) - 2)$ then $\alpha_i, \alpha_{i+1} < 0$ and we have $\beta_i, \beta_{i+1} \in]-1, 1[$ (see Figure 3.1). Recalling the definition of θ^γ and from Theorem 14 it follows the conclusion. ■

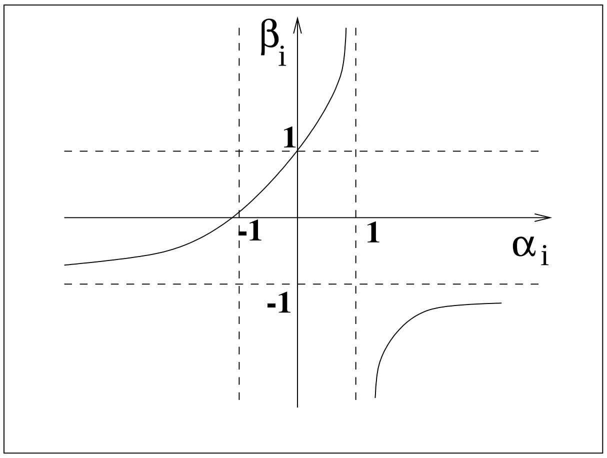


Figure 3.1

From Theorems 13 and 15, using the definitions of the times s_1^+, $s_1'^+$, t_f^+, s_1^-, $s_1'^-$, t_f^-, it follows:

Corollary 2 *Let γ be an extremal trajectory exiting the origin with control $+1$, then its first switching can occur on $\Delta_A^{-1}(0)$ only if $s_1 \neq 0$ (cond. A) or $s_1' \neq 0$ (cond. B) or $|\theta^+(t_f^+)| = \pi$ (cond. C). Moreover at most one of the conditions A, B, C holds and the corresponding time is the first switching time of γ and the first time at which γ^+ intersect $\Delta_A^{-1}(0)$. A similar result holds for γ^- and for the times s_1^-, $s_1'^-$, t_f^-.*

Referring to conditions 1, 2, 3 of Observation 1, conditions A and B correspond both to case 2 or case 1 (with $s_1^+ \neq 0$ or $s_1'^+ \neq 0$) and condition C corresponds to case 3. Moreover it is clear that for a NTAE (γ, t_1), t_1 is the first time at which γ reaches $\Delta_A^{-1}(0)$. In particular if the trajectory exits the origin with control $+1$ we have $t_1 = s_1^+$ or $t_1 = s_1'^+$ or $t_1 = t_f^+$. In the case $s_1^+ = 0$ and $s_1'^+$ not defined or viceversa and $|\theta^+(t_f^+)| < \pi$ (that implies $t_f^+ = \tau$) there are not NTAE exiting the origin with control $+1$ (an abnormal extremal exists but it never switches). This case corresponds to case 1 of Observation 1 with $s_1^+ = 0$ and s'^+ not defined or viceversa. These observations are collected in the following:

Corollary 3 *There are at most two maximal NTAE. Moreover:*

(♠) *one exits the origin with control $+1$ and its first switching is at*

- s_1^+ *iff* $s_1^+ \neq 0$;
- $s_1'^+$ *iff* $s_1^+ \neq 0$;

- t_f^+ *iff* $|\theta^+(t_f^+)| = \pi$;

(♣) *the other exits the origin with control* -1 *and its first switching is at*

- s_1^- *iff* $s_1^- \neq 0$;
- $s_1'^-$ *iff* $s_1'^- \neq 0$;
- t^- *iff* $|\theta^-(t_f^-)| = \pi$.

Finally if $|\theta^\pm(t_f^\pm)| = \pi$ *then* $\Delta_A^{-1}(0)$ *is direct at* $\gamma(t_f^\pm)$.

The following two Theorems describe the position of the switching curves of the strips whose borders are abnormal extremals.

Theorem 17 *Let* (γ, t_1) *be a NTAE,* t_i *and* t_{i+1} *two consecutive switching times,* $\mathcal{S}$ *a strip such that* $\gamma \in \partial S$ *and* U^i, U^{i+1} *two small neighborhoods of* $\gamma(t_i)$ *and* $(\gamma(t_{i+1})$. *Moreover let* U_{in}^i *and* U_{out}^i *(resp.* U_{in}^{i+1}, U_{out}^{i+1}*) be the two connected components of* $U^i \setminus \Delta_A^{-1}(0)$ *(resp.* $U^{i+1} \setminus \Delta_A^{-1}(0)$*) chosen in such a way that* γ *enters* U_{in}^i *(resp.* U_{in}^{i+1}*). Under generic conditions we have the following cases:*

(1) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ *and* $\Delta_A^{-1}(0)$ *direct at* $\gamma(t_i)$.
In this case if the switching locus of $\mathcal{S}$ *passing trough* $\gamma(t_i)$ *lies in* U_{in}^i *(resp.* U_{out}^i*) then the switching locus of* $\mathcal{S}$ *passing trough* $\gamma(t_{i+1})$ *lies in* U_{out}^{i+1} *(resp.* U_{in}^{i+1}*).*

(2) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ *and* $\Delta_A^{-1}(0)$ *inverse at* $\gamma(t_i)$.
In this case if the switching locus of $\mathcal{S}$ *passing trough* $\gamma(t_i)$ *lies in* U_{in}^i *(resp.* U_{out}^i*) then the switching locus of* $\mathcal{S}$ *passing trough* $\gamma(t_{i+1})$ *lies in* U_{in}^{i+1} *(resp.* U_{out}^{i+1}*).*

(3) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ *and* $\Delta_A^{-1}(0)$ *direct at* $\gamma(t_i)$.
In this case we have the same conclusion of case **(2)**.

(4) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ *and* $\Delta_A^{-1}(0)$ *inverse at* $\gamma(t_i)$.
In this case we have the same conclusion of case **(1)**.

Proof. Let f_i (resp. A_i) be the sign of $-\Delta_B/\Delta_A$ (resp. Δ_A) on U_{in}^i and B_i the sign of Δ_B on U^i. By hypothesis, taking U^i sufficiently small, these quantities are well defined and we have $f_i = -A_i B_i$. Moreover set $\theta_i = +1$ if $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\theta_i = -1$ if $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$.

Claim $B_{i+1} = \theta_i B_i$

<u>Proof of the Claim</u> From $sgn(\dot{\theta}^\gamma(t)) = sgn(\Delta_B(\gamma(t)))$ we have that $\bar{v}^\gamma(t)$ (see formula (1.10)) is a vector rotating counterclockwise in U^i (resp. U^{i+1}) iff $B_i > 0$ (resp. $B_{i+1} > 0$). Recalling that $\theta_i, \theta_{i+1} \in \{0, \pm\pi\}$ it is clear that B_i and B_{i+1} have the same

sign iff $\theta_{i+1} = \theta_i \pm \pi$.

Case 1 First suppose Δ_A direct at $\gamma(t_i)$. In this case from Theorem 13 we clearly have $A_{i+1} = -A_i$. Now if the switching loci of $\mathcal{S}$ lye one in U^i_{in} and one in U^{i+1}_{in} (resp. U^i_{out} and U^{i+1}_{out}) then, from Proposition 2, we have $f_i = -f_{i+1}$. This occurs iff $-A_iB_i = +A_{i+1}B_{i+1} = -A_iB_i\theta_i$, from which it follows $\theta_i = +1$.

On the other side if the switching loci of $\mathcal{S}$ lye one in U^i_{in} and one in U^{i+1}_{out} (resp. U^i_{out} and U^{i+1}_{in}) then $f_i = +f_{i+1}$ that occurs iff $\theta_i = -1$.

Case 2 If Δ_A is inverse at $\gamma(t_i)$ we have $A_{i+1} = +A_i$. Now if the switching loci of $\mathcal{S}$ lye one in U^i_{in} and one in U^{i+1}_{in} (resp. U^i_{out} and U^{i+1}_{out}) we have $f_i = -f_{i+1}$. This occurs iff $\theta_i = -1$.

On the other side if the switching loci of $\mathcal{S}$ lye one in U^i_{in} and U^{i+1}_{out} (resp. U^i_{out} and U^{i+1}_{in}) then $f_i = +f_{i+1}$ that occurs iff $\theta_i = +1$. ■

Theorem 18 *Let (γ, t_1) be a NTAE and let $\mathcal{S}^1$ and $\mathcal{S}^2$ be two stripes such that $\{\gamma\} = \mathcal{S}^1 \cap \mathcal{S}^2$. Let $\bar{t}$ be a switching time for γ and U a small neighborhood of $\gamma(\bar{t})$ such that $U \setminus \Delta_A^{-1}(0)$ has two connected components U_{in} and U_{out}, chosen in such a way γ enter U from U_{in}. Then, under generic conditions, the switching loci of $\mathcal{S}^1$ and $\mathcal{S}^2$ passing trough $\gamma(\bar{t})$ satisfy the following:*

(a) *they lye both in U_{in} or both in U_{out};*

(b) *they are tangent to supp(γ) in $\gamma(\bar{t})$.*

Proof. of (a). By the analysis of the singularities at the first switching time (see Figure 3.2 and cfr. Chapter 2) we know that **(a)** of Theorem 18 is true in the special case $\bar{t} = t_1$. Using Theorem 17 and by induction it follows the thesis.
Proof. of (b). **(b)** can be proved with an argument entirely similar to the proof of Proposition 3.3 of [17] and using induction. ■

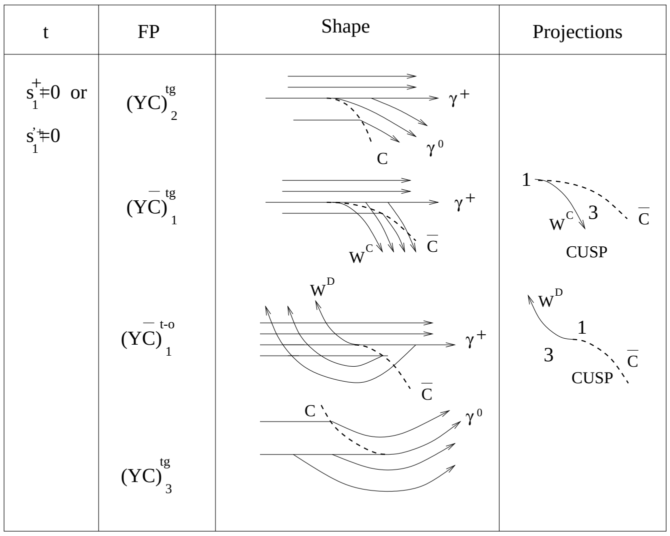


Figure 3.2

3.2 Classification of Abnormal Extremals

In Chapter 2 a complete classification of the singularities of the extremal synthesis is realized. The singularities involving abnormal extremals along $\gamma^+ \cup \gamma^-$ are collected in Figure 3.2, while for the singularities along γ^0, W^C, W^D we refer to figures 2.6, 2.7, 2.8, 2.9 of Chapter 2.

Let (γ, t_1) be a NTAE and $t_1 < t_2 ... < t_{n(\gamma)-1} < t_{n(\gamma)} := \tau$ the sequence of switching times. The previous Theorems are summarized by the rules below. **R1.** is a consequence of Theorems 17 and 18, **R2.** of Theorem 16 while **R3** is a consequence of the following obvious fact and can be obtained from figure 3.2, and figures 2.6, 2.7, 2.8, 2.9 of Chapter 2. If from the singularity at t_i ($i = 1, ..., n(\gamma) - 1$) exits an abnormal arc of kind γ^0 (resp. W^C, W^D) then in the singularity at t_{i+1} enters an abnormal arc of kind γ^0 (resp. W^C, W^D).

R1. Let $\mathcal{S}^1$ and $\mathcal{S}^2$ the two strips such that $\gamma \in \mathcal{S}^1 \cap \mathcal{S}^2$ and let $\bar{\mathcal{S}} := \mathcal{S}^1 \cup \mathcal{S}^2$. We have the following cases:

- **(1)** $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.
  In this case if the switching locus of $\bar{\mathcal{S}}$ passing trough $\gamma(t_i)$ lies in U^i_{in} (resp. U^i_{out}) then the switching locus of $\bar{\mathcal{S}}$ passing trough $\gamma(t_{i+1})$ lies in U^{i+1}_{out} (resp. U^{i+1}_{in}).
- **(2)** $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1})$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.
  In this case if the switching locus of $\bar{\mathcal{S}}$ passing trough $\gamma(t_i)$ lies in U^i_{in} (resp.

U^i_{out}) then the switching locus of $\bar{\mathcal{S}}$ passing trough $\gamma(t_{i+1})$ lies in U^{i+1}_{in} (resp. U^{i+1}_{out}).

(3) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ direct at $\gamma(t_i)$.
In this case we have the same conclusion of case **(2)**.

(4) $\theta^\gamma(t_i) = \theta^\gamma(t_{i+1}) \pm \pi$ and $\Delta_A^{-1}(0)$ inverse at $\gamma(t_i)$.
In this case we have the same conclusion of case **(1)**.

R2. If $\Delta_A^{-1}(0)$ is inverse at t_i and t_{i+1} then $\theta^\gamma(t_{i+1}) = \theta^\gamma(t_i)$.

R3. Only the following sequence of singularities are possible:

singularity at t_i	possible singularity at t_{i+1}	AA(i)
$(YC)_2^{tg}$,$(YC)_3^{tg}$,2,4,6,8,12,16,20,24	1,2,3,4,5,6,7,8	γ^0
$(Y\bar{C})_1^{tg}$,1,3,7,9,10,13,14,19,23	9,13,18,19,20,22,23,24	W^C
$(Y\bar{C})_1^{t-o}$,5,11,15,17,18,21,22	10,11,12,14,15,16,17,21	W^D

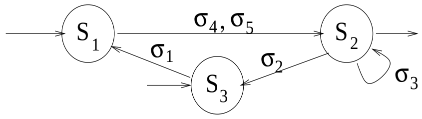
Let (γ, t_1) be a NTAE and suppose that its switching times are $t_1 < t_2 < ... < t_n = \tau$. In the following we describe all the possible sequence of singularities of γ by a *set of word* recognizable by an *Automaton*. For details on automata and languages theory we refer to [24].

Definition 27 *Let Σ be a finite set and consider the set Σ^* of ordered n–tuples $s = (\sigma_1, ..., \sigma_k)$, $\sigma_i \in \Sigma$ $(i = 1, ...k)$, $k \geq 0$. We say that $\sigma \in \Sigma$ is a* letter*, Σ is an* alphabet *and $s = (\sigma_1, ..., \sigma_k) \in \Sigma^*$ is a* word of length k.

Definition 28 *Let Σ be a finite* alphabet*, an* automaton $\mathcal{A}$ *over Σ consists of the following:*

- *a finite set of* states $\mathbb{S} = \{S^1, ..., S^n\}$*;*
- *a set of* initial states $\mathbb{I} \subseteq \mathbb{S}$*;*
- *a set of* terminal states $\mathbb{T} \subseteq \mathbb{S}$*;*
- *a set of* edges *that is a subset $\mathbb{E} \subseteq \mathbb{S} \times \Sigma \times \mathbb{S}$. An* edge *is indicated as (S_1, σ, S_2) and we say that it begins at S_1, it ends at S_2 and it carries the label σ.*

Usually an *automaton* is represented by a set of circles (states) and a set of arrows that connect the circles (the edges). The initial (resp. final) states are labelled by arrows pointing toward (resp. away from) the circle. If there are several edges beginning and ending at the same states they are replaced by a single arrow carrying several labels.



A *path* in $\mathcal{A}$ is a finite sequence of edges of the type $(S_1, \sigma_1, S_2)(S_2, \sigma_2, S_3)...(S_k, \sigma_k, S_{k+1})$. If $S_1 \in \mathbb{I}$ and $S_{k+1} \in \mathbb{T}$ we say that the *path* is successful.

Definition 29 *A set of words $\Omega \subset \Sigma^*$ is said to be recognizable by $\mathcal{A}$ if for every word $(\sigma_1, \sigma_2, ..., \sigma_m) \in \Omega$ of length m there exists $S_1, ..., S_{m+1} \in \mathbb{S}$ such that:*

- *$(S_i, \sigma_i, S_{i+1}) \in \mathbb{E}$ for every $i = 1, ..., m$;*
- *$(S_1, \sigma_1, S_2)(S_2, \sigma_2, S_3)...(S_m, \sigma_m, S_{m+1})$ is a successful path.*

To prove Theorem 12 amounts precisely to construct an *automaton* describing all the possible sequences of singularities along a NTAE. First we build an automaton, naturally associated to a system, with the simplest possible set of edges. By this automaton we can prove that the ribbon singularity is realized, but more than one sequence of singularities may correspond to a recognizable word. Then we describe how to build a more complicated automaton that has the required property that is to every recognizable word it corresponds one and only one sequence of singularities.

For us the set of *states* is the set of the 28 singularities:

$$\begin{aligned}\mathbb{S} := \{&(YC)_2^{tg}, (Y\bar{C})_1^{tg}, (Y\bar{C})_1^{t-o}, (YC)_3^{tg},\\ &1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}\end{aligned}$$

and the alphabet is

$$\mathbb{E} := \{0, \pi\}$$

that is (if we are considering two singularities at t_i and t_{i+1}) the set of values assumed by the function $\Delta\theta_i^\gamma := |\theta^\gamma(t_{i+1}) - \theta^\gamma(t_i)|$. The set of *initial states* is constituted by the singularities $(YC)_2^{tg}, (Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}, (YC)_3^{tg}$ and the set of *terminal states* coincides with $\mathbb{S}$. Using rules **R1**$\div$ **R3** it is easy to write Table A that show how the *edges* connect the *states*. For example from the *state* (singularity) 18, and the *letter* π we may reach the states $14, 15, 16$. This means that the edges of $\mathbb{E}$ that begin at the state 18 are: $(18, \pi, 14), (18, \pi, 15), (18, \pi, 16)$. It is clear that for this automaton every word is recognizable. From Table A we have:

Theorem 19 *All the states $1 \div 24$ can be reached with at most two edges. More precisely only the singularity 17 need in fact two edges. Moreover the singularity number 22 (the ribbon) can be realized with the edge $((Y\bar{C})_1^{tg}, \pi, 22)$.*

$letters \rightarrow$ $states$ $\downarrow$	0	π
$(YC)_2^{tg}$	1,2,5,6	3,4,7,8
$(Y\bar{C})_1^{tg}$	9,18,19,20	13,22,23,24
$(Y\bar{C})_1^{t-o}$	14,15,16,21	10,11,12
$(YC)_3^{tg}$	1,2,5,6	3,4,7,8
1	13,22,23,24	9,18,19,20
2	3,4,7,8	1,2,5,6
3	9,18,19,20	13,22,23,24
4	1,2,5,6	3,4,7,8
5	10,11,12,17	14,15,16
6	1,2,5,6	3,4
7	13,22,23,24	9
8	3,4,7,8	1,2
9	13,22,23,24	9,18,19,20
10	13,22,23,24	9,18,19,20
11	14,15,16,21	10,11,12,17
12	3,4,7,8	1,2,5,6
13	9,18,19,20	13,22,23,24
14	9,18,19,20	13,22,23,24
15	10,11,12,17	14,15,16,21
16	1,2,5,6	3,4,7,8
17	10,11,12,17	14,15,16
18	10,11,12,17	14,15,16
19	9,18,19,20	13
20	1,2,5,6	3,4
21	14,15,16,21	10,11,12
22	14,15,16,21	10,11,12
23	13,22,23,24	9
24	3,4,7,8	1,2

Table A

In Figure 3.3 it is shown the automaton with all the states, but only the edges of the kind $((YC)_2^{tg}, \ . \ , \ . \ )$, $((Y\bar{C})_1^{tg}, \ . \ , \ . \ )$, $(15, \ . \ , \ . \ )$.

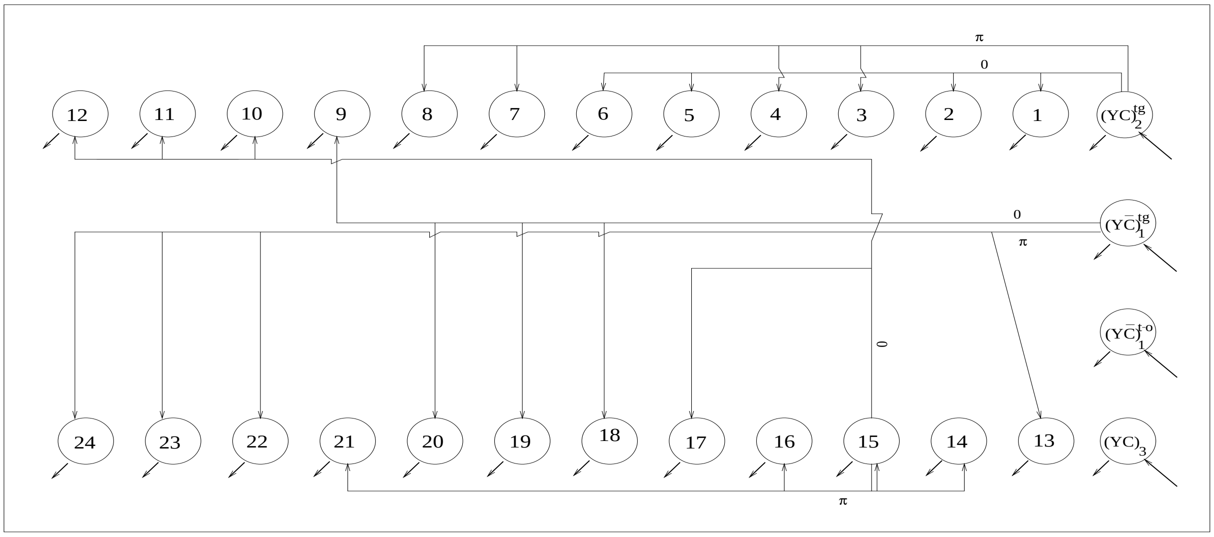


Figure 3.3: The automaton (some edges are omitted)

To describe in a unique way the set of singularities, with a set of words recognizable by an automaton, we need to include more information in the alphabet. First of all we need to give a name to the entry arrows (e.g. I_1, I_2, I_3, I_4, corresponding respectively to the singularities $(YC)_2^{tg}$, $(Y\bar{C})_1^{tg}$, $(Y\bar{C})_1^{t-o}$, $(YC)_3^{tg}$). Then we have to introduce more information in the letters (i.e. we need a bigger alphabet). One way is to include the following data relatively to the next singularity:

- if $\Delta_A^{-1}(0)$ is direct or inverse (that we indicate with the letters D and I in the following),
- the kind of exiting abnormal arc (i.e. γ^0, W^C, W^D).

In this way for the new automaton we have $\mathbb{S}' = \mathbb{S}$ and $\Sigma' = \{0, \pi\} \times \{D, I\} \times \{\gamma^0, W^C, W^D\}$. Every element of Σ' is indicated as a triple $(\, . \, , \, . \, , \, . \,)$. In Table B it is written how the edges connect the states. Clearly for this new automaton to every word it corresponds a unique sequence of singularities, but not all the words that can be constructed with the alphabet are recognizable. For example the word $I_2(\pi, I, W^D)(0, D, W^C)$ is recognizable and it corresponds to the sequence of singularities $(Y\bar{C})_1^{tg} \to 22 \to 14$ while the word $I_1(0, I, W^C)$ is not recognizable. Theorem 12 is proved.

letters→ states ↓	$(0,D,\gamma^0)$	$(0,D,W^C)$	$(0,D,W^D)$	$(0,I,\gamma^0)$	$(0,I,W^C)$	$(0,I,W^D)$
$(YC)_2^{tg}$	2	1	–	6	–	5
$(Y\bar{C})_1^{tg}$	–	9	–	20	19	18
$(Y\bar{C})_1^{t-o}$	16	14	15	–	–	21
$(YC)_3^{tg}$	2	1	–	6	–	5
1	–	13	–	24	23	22
2	4	3	–	8	7	–
3	–	9	–	20	19	18
4	2	1	–	6	–	5
5	12	10	11	–	–	17
6	2	1	–	6	–	5
7	–	13	–	24	23	22
8	4	3	–	8	7	–
9	–	13	–	24	23	22
10	–	13	–	24	23	22
11	16	14	15	–	–	21
12	4	3	–	8	7	–
13	–	9	–	20	19	18
14	–	9	–	20	19	18
15	12	10	11	–	–	17
16	2	1	–	6	–	5
17	12	10	11	–	–	17
18	12	10	11	–	–	17
19	–	9	–	20	19	18
20	2	1	–	6	–	5
21	16	14	15	–	–	21
22	16	14	15	–	–	21
23	–	13	–	24	23	22
24	4	3	–	8	7	–

Table B (first part)

letters→ states ↓	(π,D,γ^0)	(π,D,W^C)	(π,D,W^D)	(π,I,γ^0)	(π,I,W^C)	(π,I,W^D)
$(YC)_2^{tg}$	4	3	–	8	–	7
$(Y\bar{C})_1^{tg}$	–	13	–	24	23	22
$(Y\bar{C})_1^{t-o}$	12	10	11	–	–	–
$(YC)_3^{tg}$	4	3	–	8	7	–
1	–	9	–	20	19	18
2	2	1	–	6	–	5
3	–	13	–	24	23	22
4	4	3	–	8	7	–
5	16	14	15	–	–	–
6	4	3	–	–	–	–
7	–	9	–	–	–	–
8	2	1	–	–	–	–
9	–	9	–	20	19	18
10	–	9	–	20	19	18
11	12	10	11	–	–	17
12	2	1	–	6	–	5
13	–	13	–	24	23	22
14	–	13	–	24	23	22
15	16	14	15	–	–	21
16	4	3	–	8	7	–
17	16	14	15	–	–	–
18	16	14	15	–	–	–
19	–	–	–	–	13	–
20	4	3	–	–	–	–
21	12	10	11	–	–	–
22	12	10	11	–	–	–
23	–	9	–	–	–	–
24	2	1	–	–	–	–

Table B (second part)

3.3 An Example

An example of system that contains a ribbon is the following (choosing properly the functions $F_1(x_2)$ and $F_2(x_2)$):

$$\begin{cases} \dot{x}_1 = \frac{1}{2}\left((1+u)+(1-u)\left(-1+\frac{3}{2}x_1+F_1(x_2)\right)\right) \\ \dot{x}_2 = \frac{1}{2}(1-u)\left(\frac{1}{2}x_1(x_1-1)(x_1-\frac{5}{4}\right)+F_2(x_2)\ . \end{cases}$$

In particular it is generated at the second switching time from the abnormal extremal exiting the origin with control +1 (see figure 3.4).

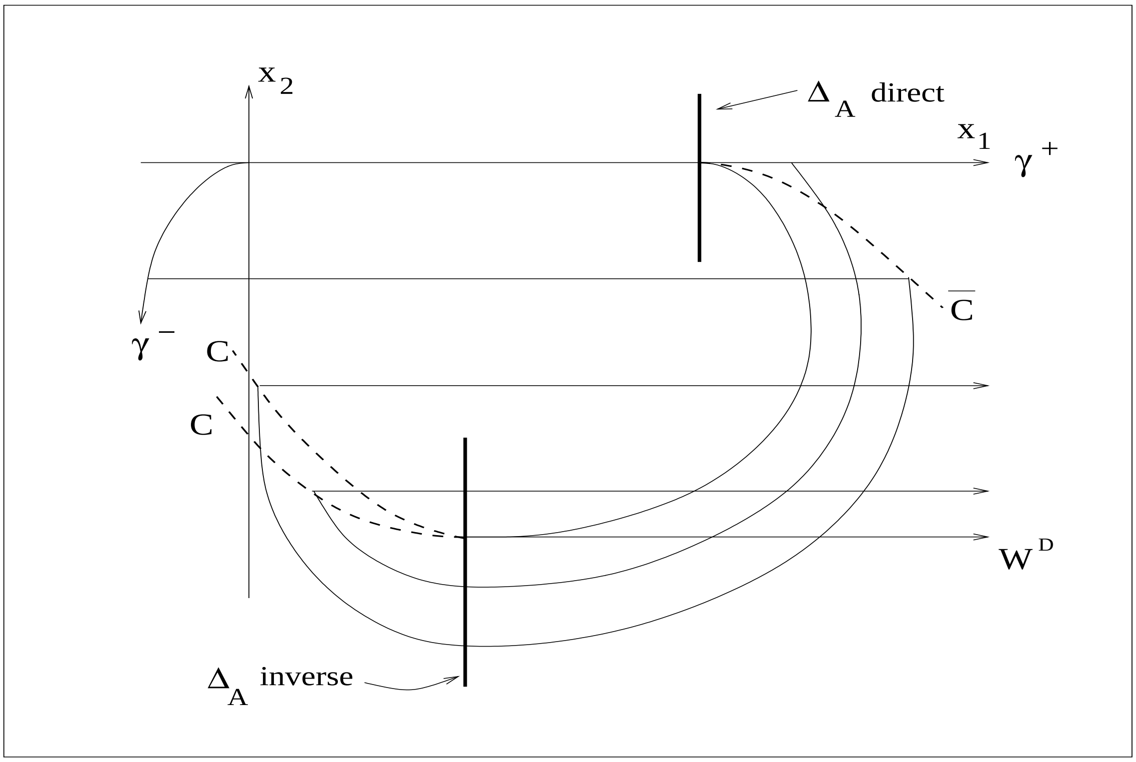


Figure 3.4: An example of synthesis involving a *ribbon* singularity

Chapter 4

The Initial Conditions Set

For fixed $\tau > 0$, in this Chapter we are interested in studying the set:

$N_0 = \{(x,p) \in \mathbb{R}^2 \times S^1$: there exists a time $t \in [0,\tau]$ and an extremal pair $(\gamma,\lambda) : [0,t] \to \mathbb{R}^2 \times S^1$ such that $\gamma(0) = 0$, $\gamma(t) = x$, $\lambda(0) = p\}$,

relatively the minimum time stabilization to the origin for the control system:

$$\dot{x} = F(x) + uG(x), \qquad x \in \mathbb{R}^2, \qquad u \in [-1,1], \qquad F(0) = 0. \tag{4.1}$$

We show that the lift of the extremal trajectories after the first switching point are the iso–surfaces of N_0 and that (x,p), (x',p) belong to N_0 ($x \neq x'$) only if x and x' are reached by the same extremal trajectory (at different times) or they belong to a "flat" region determined by singular trajectories (see Theorem 20 and Lemma 1). Using this argument we can determine some conditions under which N_0 is a piecewise $\mathcal{C}^1$ manifold (with boundary). More details are given for N_0 restricted to a neighborhood of the origin.

4.1 The locus N_0 in a neighborhood of the origin

Consider the projector $\pi : (x,p) \in \mathbb{R}^2 \times S^1 \mapsto x \in \mathbb{R}^2$. Clearly we have $\mathcal{R}(\tau) = \pi(N_0)$. If we set $\Lambda^0(x) = \{p \in S^1 : (x,p) \in N_0\}$ we can regard N_0 as a trivial fiber bundle with base $\mathcal{R}(\tau)$ and fiber $\Lambda^0(x)$ i.e. $N_0 = \bigcup_{x \in \mathcal{R}(\tau)} (x, \Lambda^0(x))$. Obviously we have $(x, \Lambda^0(x)) = \pi^{-1}(x) \cap N_0$.

The structure of N_0 is contained in the following Theorem (see Chapter 1 for the definitions of the function $\theta^+(.)$ and of the times $t_f^+, t_i^+, t_i'^+, s_i^+, s_i'^+$):

Theorem 20 *Chose a coordinate system such that $G(0) = (0,1)$.*

A) *Let $x \in \mathcal{R}(\tau) \setminus (supp(\gamma^+) \cup supp(\gamma^-))$ and fix an extremal trajectory that reaches x at some time $\bar{t}$:*

$$\gamma : [0,\bar{t}] \to \mathcal{R}(\tau)$$

$$\gamma(0) = 0, \quad \gamma(\bar{t}) = x.$$

Then there exists a unique covector $\lambda : [0, \bar{t}] \to \mathbb{R}^2$ *s.t.* $|\lambda(0)| = 1$ *and* (γ, λ) *is an extremal pair in* $[0, \bar{t}]$*. Moreover, let* $t_0(\gamma) \in]0, t_f^+[$ *be the first switching time of* γ*, say from constant control +1 to constant control -1 or* $\varphi(\gamma(t))$*, then we have:*

$$\Lambda^0(x) = \begin{cases} \theta^+(t_0(\gamma)) & \text{if } t_0(\gamma) \in [s_i^+, t_i^+] \\ \pi + \theta^+(t_0(\gamma)) & \text{if } t_0(\gamma) \in [s_i'^+, t_i'^+] \end{cases} \tag{4.2}$$

B) $\Lambda^0(0) = [-\pi, \pi]$

C) *Let* $x \in$ *supp*$(\gamma^+) \setminus \{0\}$ *and let* $t(x) \in]0, t_f^+]$ *be the first time such that* $x = \gamma^+(t(x))$*, then:*

$$\Lambda^0(x) = \begin{cases} [\theta^+(t(x)), \pi + \theta^+(t_{i-1}'^+)] & \text{if } t(x) \in]s_i^+, t_i^+] \\ [\theta^+(t_i^+), \pi + \theta^+(t_{i-1}'^+)] & \text{if } t(x) \in]t_i^+, s_i'^+] \\ [\theta^+(t_i^+), \pi + \theta^+(t(x))] & \text{if } t(x) \in]s_i'^+, t_i'^+] \\ [\theta^+(t_i^+), \pi + \theta^+(t_i'^+)] & \text{if } t(x) \in]t_i'^+, s_{i+1}^+] \end{cases}$$

Proof Let us first prove **A)**. By definition p exists. We have to prove the uniqueness. Since γ is extremal, we know that there exists a non trivial covector $\lambda : [0, \bar{t}] \to \mathbb{R}^2$ s.t. for a.e. $t \in [0, t_0]$:

$$\dot{\lambda}(t) = -\lambda(t) \cdot \nabla Y(\gamma(t))$$
$$\lambda(t) \cdot G(\gamma(t)) \geq 0.$$

Using the definition of v^+ (see Chapter 1) and by the property $\lambda(t) \cdot v^+(v_0, t_0; t) =$const (for $t \in [0, t_0]$, see [17]), PMP implies that for a.e. $t \in [0, t_0]$:

$$\begin{aligned} 0 &\leq \lambda(t) \cdot G(\gamma(t)) = \\ &= \lambda(t) \cdot v^+(G(\gamma(t)), t; t) = \\ &= \lambda(0) \cdot v^+(G(\gamma(t)), t; 0) = \\ &= \lambda(0) \cdot \bar{v}^+(t). \end{aligned} \tag{4.3}$$

Now $\bar{v}^+(t)$ is a vector that satisfy $\arg(\bar{v}^+(0), \bar{v}^+(t)) = \theta^+(t)$, where θ^+ is a function whose range reaches π exactly at time t_f^+ (see Chapter 1). So with our local coordinates we have the situation of the following figure:

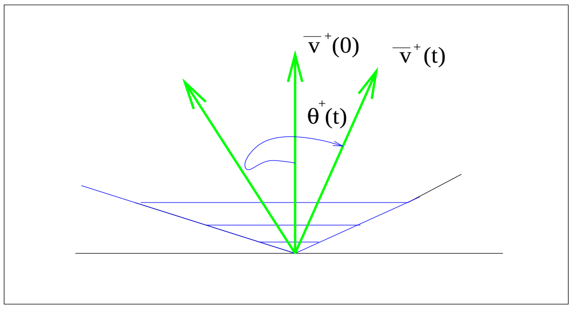


Figure 4.1: Proof of Theorem 20

All the $\lambda(0)$ in the sketched region satisfy the condition 4.3. Now

- if $t_0 \in [s_i^+, t_i^+[$ (resp. $t_0 \in [s_i'^+, t_i'^+[$):

  1. from Theorem 1 (Chapter 1) at time t_0 we have a YX switching;
  2. $\dot{\theta}^+(t_0) > 0$ (resp. < 0); (4.4)
  3. formula (4.3) is satisfied with the equality at time t_0 i.e. $\lambda(0) \cdot \bar{v}^+(t_0) = 0$ (4.5)

- if $t_0 = t_i^+$ (resp. $t_i'^+$) then

  1. from Theorem 1 (Chapter 1) at time t_0 we have a YX switching;
  2. θ has a maximum (resp. a minimum) at this time; (4.6)
  3. $\lambda(0) \cdot \bar{v}^+(t_0) = 0$. (4.7)

Notice that Theorem 1 (Chapter 1) excludes that $t_0 \in]t_i^+, s_i'^+[$ or $t_0 \in]t_i'^+, s_{i+1}^+[$. Formulas (4.4, 4.5) in the first case and (4.6), (4.7) in the second select exactly the $\lambda(0)$ given by the right hand side of formula (4.2) that includes both cases. Now, since the evolution equation of λ (see **i)**, Chapter 1) satisfies the Caratheodory conditions, $\lambda(t)$ is unique, and it follows (4.2).

Because of the request "a.e." in the PMP, in the origin every $\lambda(0)$ is allowed. By normalizing we have $\Lambda^0(0) = [-\pi, \pi]$. this proves **B)**.

By the same argument of the proof of formula (4.2) we have that for a.e. $t \in]0, t(x)]$ PMP says that:

$$0 \leq \lambda(0) \cdot \bar{v}^+(t) \tag{4.8}$$

Again all the $\lambda(0)$ in the sketched region of figure 4.1 satisfy this condition. From the definition of the times $t_i^+, s_i^+, t_i'^+, s_i'^+$ it follows the expression for $\Lambda^0(x)$ in the case $x \in \mathrm{supp}(\gamma^+) \setminus \{0\}$, that concludes the proof. ■

<u>*Observation*</u> **2** A similar Theorem can be written for the trajectory γ^- that starts from the origin with control -1.

Let $I(0)$ be a small neighborhood of the origin. The shape of $N_0|_{I(0)}$ can be obtained from Theorem 20 and it is drawn in figure 4.2.

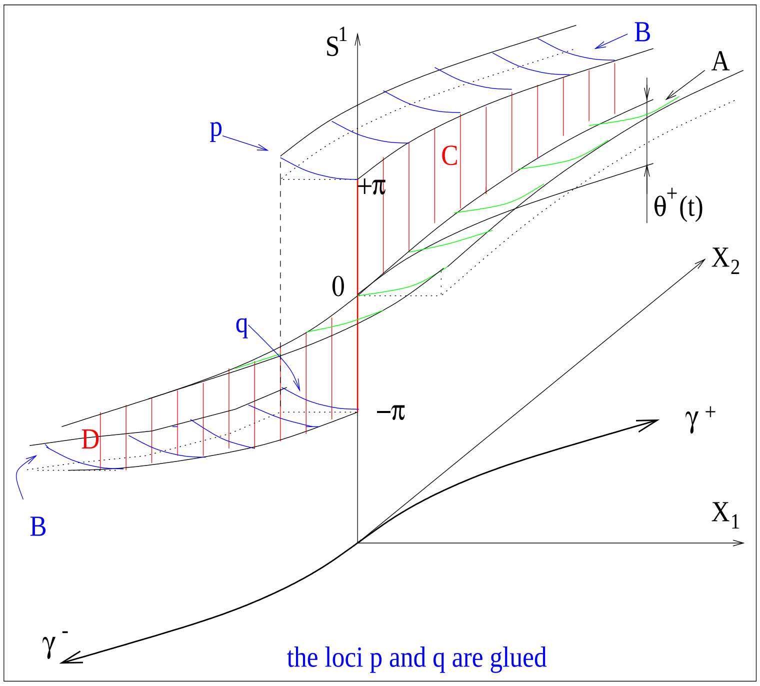


Figure 4.2: N_0 in a neighborhood of the origin

Now we will describe a piecewise $\mathcal{C}^1$ diffeomorphism from $N_0|_{I(0)}$ to $R := U \setminus B(0,1) \mapsto \mathbb{R}^2$ where U is a connected open region of $\mathbb{R}^2$ containing an open ball with center in 0 and radius 1 (see figure 4.3). This proves that $N_0|_{I(0)}$ is a piecewise $\mathcal{C}^1$ manifold with boundary. In the following the subscript r will indicate the restriction to $I(0)$.

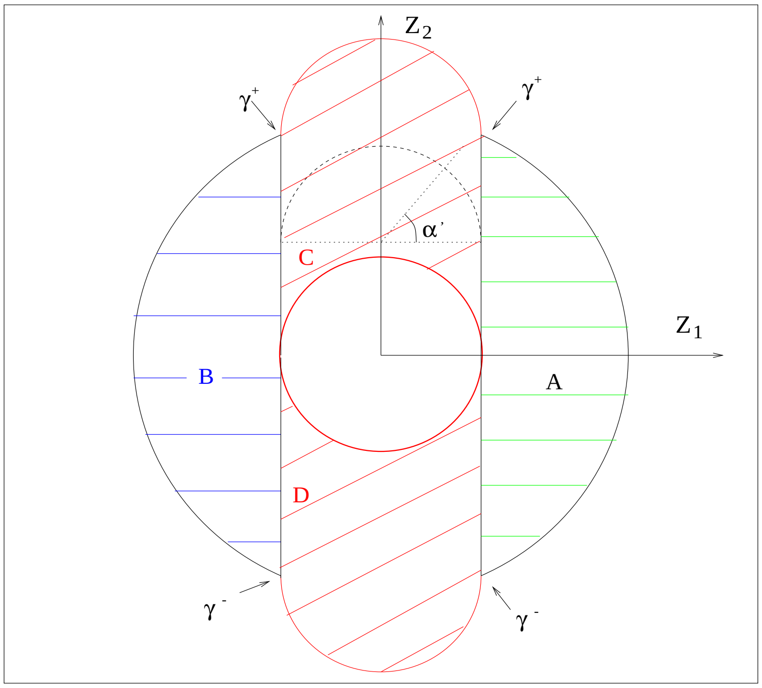


Figure 4.3: The set R

Chose a local system of coordinates such that $G(0)=(0,1)$ and let $x_1^+(x_2)$ (resp. $x_1^-(x_2)$) the support of γ_r^+ (resp. γ_r^-) expressed as a function of x_2. Moreover let (z_1, z_2) be the Cartesian coordinates of the plane containing R. Call O the region of N_0 identified by $\pi(O)=0$. Any point $y \in O$ is uniquely identified by an angle $\alpha \in [-\pi, \pi]$. Consider the map $\xi_O : O \to R$ defined by:

$$\alpha \mapsto (z_1(\alpha), z_2(\alpha)) = (\cos\alpha, \sin\alpha). \tag{4.9}$$

Let U_A and U_B the two connected components of $I(0) \setminus (\text{supp}(\gamma_r^+) \cup \text{supp}(\gamma_r^-))$, chosen is such a way U_A lies on the right of γ^+. Call A the region of N_0 identified by $\pi(A) = U_A$. Any point $y \in A$ is uniquely identified by its projection $x = \pi(y)$. Let $(x_1,\ x_2)$ the Cartesian coordinates of $x \in U_A$. Consider the map $\xi_A : A \to R$ defined by:

$$\begin{aligned}(x_1, x_2) \ \ &\mapsto\ \ (z_1(x_1,x_2), z_2(x_1,x_2)) = \\ &= \begin{cases} (x_1 + 1 - x_1^-(x_2), x_2) & \text{if } \ x_2 \le 0 \\ (x_1 + 1 - x_1^+(x_2), x_2) & \text{if } \ x_2 \ge 0 \end{cases}.\end{aligned}$$

Call C the region of N_0 identified by $\pi(C) =$ supp $(\gamma_r^+) \setminus \{0\}$. Any $y \in C$ is uniquely identified by a couple (x, α) where $x \in$ supp$(\gamma_r^+) \setminus \{0\}$ and $\alpha \in [\theta^+(t), \pi]$. Let $t(x)$ the time s.t. $x = \gamma_r^+(t(x))$. Consider the map $\xi_C : C \to R$ defined by:

$$\begin{aligned}&(x, \alpha) \mapsto (z_1(x,\alpha), z_2(x,\alpha)) = (\cos\alpha', x_2 + \sin\alpha') \\ &\text{where } \alpha' = \frac{\pi}{\pi - \theta^+(t(x))}(\alpha - \theta^+(t(\ x)).\end{aligned}$$

In similar way we can define the maps ξ_B, ξ_D. The map:

$$\xi(y) = \begin{cases} \xi_O(y) & \text{if } y \in O \\ \xi_A(y) & \text{if } y \in A \\ \xi_B(y) & \text{if } y \in B \\ \xi_C(y) & \text{if } y \in C \\ \xi_D(y) & \text{if } y \in D \end{cases} \tag{4.10}$$

is a piecewise $\mathcal{C}^1$–diffeomorphism of $N_0|_{I(0)} \to R$.

<u>*Observation*</u> **3** In the interior of every region this diffeomorphism is $\mathcal{C}^\infty$. N_0 looses to be $\mathcal{C}^1$ on the $A-C$, $\ C-B$, $\ A-D$ and $D-B$ edges that on R correspond all to $\gamma_r^+ \cup \gamma_r^-$.

<u>*Observation*</u> **4** Notice that N_0 and N are piecewise–$\mathcal{C}^1$ diffeomorphic in a neighborhood of the lift of origin. Moreover they have the same projection on $\mathbb{R}^2$.

4.2 Intersection of extremal pairs in the cotangent bundle

Define the loci:

$\mathcal{K}_1 = \{(x,p) \in N_0$: there exist two times $t_1 < t_2 \leq \tau$ and an extremal pair (γ, λ) such that $\gamma(t_1) = \gamma(t_2) = x$, $\lambda(0) = p\}$,

$\mathcal{K}_2 = \{(x,p) \in N_0$: $\exists\ \varepsilon > 0$, three times $t_0 < t_1 \leq t_2 \leq \tau$ and two extremal pairs (γ_1, λ_1), (γ_2, λ_2) such that $\gamma_1(t_1) = \gamma_2(t_2) = x$, $\gamma_1|_0^{t_0} = \gamma_2|_0^{t_0}$, $\lambda_1(0) = \lambda_2(0) = p$, the trajectory coinciding with γ_1 (and γ_2) in the interval $[0, t_0]$ and corresponding to the singular control φ in $[t_0, t_0 + \varepsilon[$ is extremal$\}$.

Note that $\mathcal{K}_1$ and $\mathcal{K}_2$ are not necessarily disjoint. Moreover we can have $\mathcal{K}_1 = \emptyset$ or $\mathcal{K}_2 = \emptyset$. In particular both of them are empty if τ is sufficiently small. We have the following:

Lemma 1 *Let x_1, x_2 be two points of $\mathcal{R}(\tau)$, then (x_1, p) and (x_2, p) belong to N_0 iff one of the following conditions holds:*

C1 *there exists an extremal pair (γ, λ) s.t. $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$, $\lambda(0) = p$ for some t_1, $t_2 \leq \tau$;*

C2 *there exist $\varepsilon > 0$, three times t_0, t_1, $t_2 \leq \tau$, $(t_0 < t_1$, $t_0 < t_2)$ and two extremal pairs (γ_1, λ_1), (γ_2, λ_2) such that $\gamma_1(t_1) = x_1$, $\gamma_2(t_2) = x_2$, $\lambda_1(0) = \lambda_2(0) = p$, $\gamma_1|_0^{t_0} = \gamma_2|_0^{t_0}$, the trajectory coinciding with γ_1 (and γ_2) in the interval $[0, t_0]$ and corresponding to the singular control φ in $[t_0, \varepsilon[$ is extremal.*

Proof The proof that the conditions are sufficient is obvious. Now we will prove that they are necessary. Let $(\gamma_1, \lambda_1) : [0, t_1] \to \mathbb{R}^2 \times \mathbb{R}_2$ and $(\gamma_2, \lambda_2) : [0, t_2] \to \mathbb{R}^2 \times \mathbb{R}_2$ be the extremal trajectories such that $\gamma_1(t_1) = x_1$, $\gamma_2(t_2) = x_2$, $\lambda_1(0) = \lambda_2(0) = p$. By Theorem 20 and from the strict monotonicity of θ^+ in the intervals $[s_i^+, t_i^+[$ and $[s_i'^+, t_i'^+[$ (that follows from the generic assumption of [17]) we can conclude that:

1) if γ_1 and γ_2 switch at least one time then (x_1, p), $(x_2, p) \in N_0$ implies that γ_1 and γ_2 switch the first time at the same point. **C1** corresponds to the case in which they are reached by the same extremal trajectory. **C2** is the only possible case in which there is not forward uniqueness for the optimal feedback (see [17]);

2) if γ_1 has no switching and γ_2 has at least one switching then $\gamma_2|_{[0,t_1]} = \gamma_1$ and **C1** holds;

3) if γ_1 and γ_2 have no switching then either γ_1 is the restriction of γ_2 or viceversa, so we are in case **C1**.

This concludes the proof. ■

Theorem 21 *Let (x,p) belong to N_0, then the extremal pair (γ,λ) such that $\gamma(t) = x,\quad \lambda(t) = p$ for some $t \leq \tau$ is unique iff $(x,p) \notin \mathcal{K}_1 \cup \mathcal{K}_2$*

Proof By definition of the loci $\mathcal{K}_1$ and $\mathcal{K}_2$, if (γ,λ) is unique then $x \notin \mathcal{K}_1 \cup \mathcal{K}_2$. Now we will prove the other implication. Assume that there exist two distinct extremal pairs (γ_1,λ_1), (γ_2,λ_2) such that $\gamma_1(t_1) = \gamma_2(t_2) = x$ for some $t_1,\ t_2$ and $\lambda_1(0) = \lambda_2(0) = p$. With arguments similar to the proof of Lemma 1 we can conclude that $x \in \mathcal{K}_1 \cup \mathcal{K}_2$ (in this case we have $x_1 = x_2$ but two different extremal pairs; more precisely if **C1** (resp. **C2**) holds then $x \in \mathcal{K}_1$ (resp. $x \in \mathcal{K}_2$)). ■

4.3 The lift of N_0 in the (x,p,t) space.

Consider the following lift of N_0 in the (x,p,t) space:

$$\begin{aligned}\bar{N}_0 = &\{(x,p,t) \in \mathbb{R}^2 \times S^1 \times [0,\tau] : \text{there exists an extremal pair } (\gamma,\lambda) :\ [0,t] \to \mathbb{R}^2 \times \mathbb{R}_2 \\ &\text{such that } \gamma(0) = 0,\ \gamma(t) = x,\ \lambda(0) = p\}.\end{aligned}$$

We have the following:

Lemma 2 *Let (x,p,t) belong to $\bar{N}_0$, then, generically, the extremal pair (γ,λ) such that $\gamma(t) = x$, $\lambda(t) = p$ is unique.*

Proof By Theorem 21 we may consider only the regions $\mathcal{K}_1$ and $\mathcal{K}_2$. If $x \in \mathcal{K}_1$ then **C1** is satisfied but with $t_1 \neq t_2$. If $x \in \mathcal{K}_2$ we may assume that t_1 and t_2 are different by genericity. ■

By this Lemma it is clear that, generically, in the (x,p,t) space no extremal trajectories intersect each other. In fact a more detailed local analysis (similar to the one of section 4.1) shows that holds:

Theorem 22 *Generically $\bar{N}_0$ is a piecewise $\mathcal{C}^1$ manifold with boundary.*

Now Theorem 21 and Lemma 2 show that in fact the projection $\bar{N}_0 \to N_0$ is singular only on the set $\mathcal{K}_1 \cup \mathcal{K}_2$. But $\mathcal{K}_2$ is a subset of a "flat region" because of the following:

Lemma 3 *Let $(x,p) \in \mathcal{K}_2$ and assume that there exists $\varepsilon > 0$ such that $\gamma_1|_{[t_0,t_0+\varepsilon]}$, $\gamma_2|_{[t_0,t_0+\varepsilon]}$, correspond to the singular control φ then there exists $\delta > 0$ such that for every $(x',p') \in\ N_0$ satisfying $|p'-p| < \delta$ and $|x'-x| < \delta$ we have $p' = p$.*

Proof Since $(x,p) \in \mathcal{K}_2$ we know that there exists a time t_0 such that $\gamma_1|_0^{t_0} = \gamma_2|_0^{t_0}$ and the trajectory coinciding with γ_1 (and γ_2) in the interval $[0,t_0]$ and corresponding to the singular control φ in $[t_0, t_0+\varepsilon[$ is extremal. Let $\bar{t}_0$ be the first time such that these conditions are satisfied. We may choose ε small enough such that γ_1 (and γ_2) corresponds to a constant control , say $+1$, in $]\bar{t}_0-\varepsilon, \bar{t}_0[$ Consider the locus:

$\mathcal{F}(\bar{t}_0) = \{(x,p) \in N_0 :\ \exists\, \varepsilon > 0$, a time t ($\bar{t}_0 \le t \le \tau$) and an extremal pair (γ,λ) such that $\gamma(t) = x$, $\lambda(0) = p$ and corresponding to constant control $+1$ in $]\bar{t}_0-\varepsilon, \bar{t}_0]$; moreover the trajectory coinciding with γ in the interval $[0,\bar{t}_0]$ and corresponding to the singular control φ in $[\bar{t}_0, \bar{t}_0+\varepsilon[$ is extremal$\}$.

It is clear that (x_1,p_1), $(x_2,p_2) \in \mathcal{F}(\bar{t}_0)$ implies $p_1 = p_2$. From the analysis [32] it follows that generically $\mathcal{F}(\bar{t}_0)$ is a connected piecewise $\mathcal{C}^1$ manifold with boundary and it easy to check that the above assumptions implies $(x,p) \in \mathrm{int}(\mathcal{F}(\bar{t}_0))$. This conclude the proof. ■

Analyzing the boundary of $\mathcal{K}_2$ one get the following:

Theorem 23 *$N_0 \setminus \mathcal{K}_1$ is a piecewise $\mathcal{C}^1$ manifold with boundary.*

Note that if there are not overlaps then $\mathcal{K}_1 = \emptyset$.

Chapter 5

Morse Properties for the Minimum Time Function

5.1 Introduction

In this Chapter we consider the following geometrical problem. Given a smooth manifold M and two smooth vector fields X and Y on M, we want to steer a point $p \in M$ to a point $q \in M$ in minimum time using only integral curves of the two vector fields X and Y. Even if there exists such a trajectory going from p to q, it may happen that the minimum time is obtained only by a trajectory γ whose velocity $\dot{\gamma}(t)$ belongs to the segment joining $X(\gamma(t))$ and $Y(\gamma(t))$ (not being an extremum of it). Thus we first convexify the set of velocities letting $V(x) = \{vX(x) + (1-v)Y(x) : 0 \leq v \leq 1\}$ be the set of admissible velocities at a generic point x of M. If we fix the point p and let q vary on M we define the function $\mathbf{T}_p(q)$ as the minimum time to reach the point q from p. We are interested in investigating the generic properties of the function $\mathbf{T}_p$.

We consider the case in which M is two dimensional and the point p is an equilibrium point, that is $0 \in V(p)$. Even if the space of all admissible trajectories is infinite dimensional, it happens that the time optimal trajectories are indeed finite concatenations of X trajectories, Y trajectories and singular trajectories that correspond to a smooth vector field defined in terms of the Lie Bracket of X and Y. Moreover, the number of arcs is bounded on compact sets. It is known that a function that is the minimum of a finite number of smooth functions in generic position is a Morse function in topological sense [2, 29]. We give a positive answer to the question of V.I. Arnold: is the function $\mathbf{T}_p$ generically a Morse function in topological sense? (see Theorem 24). This result is in the same spirit of [2, 29], being now the minimization taken over a finite dimensional set of functions.

In order to prove Theorem 24 (see below), we analyze the Minimum Time Fronts (briefly MTF) that are the level sets of the minimum time function $\mathbf{T}_p$. The generic properties of this front are resumed in Theorem 25 of Section 5.2.

One can define the extremal front in an obvious way. It is interesting to notice that the extremal front at some singular points of the optimal one, generically develops only standard singularities as two dimensional wave front, see [3, 4]. Moreover, these singularities correspond exactly to projection singularities, of the set of lifted trajectories, of cusp type see Chapter 2.

The generic properties of the minimum time front on points reached by abnormal extremals (that is trajectories with vanishing Hamiltonian) are studied. By a local analysis we show that at these points the minimum time front is always tangent to the abnormal extremals, see Theorem 26. This fact was remarked also in [8] and in [41] for a quadratic cost.

The set of points q that can be reached by a trajectory starting from p in time less than or equal to $T > 0$ is called the reachable set from p in time T. The topology of this set generically changes only at a discrete set of times when a special curve of points, called overlap curve and reached by two different optimal trajectories, either starts or ends. See Theorem 27.

Finally, we study the case in which $M = \mathbb{R}^2$ and p is replaced by a smooth two dimensional manifold with boundary. In this case, under a local controllability assumption, we can also prove the semiconcavity of the minimum time function. This property is typically encountered in calculus of variation problems or optimal control problems with set of velocities $V(x)$ of the same dimension of the state space, see [21] and references therein. (Notice that, under generic conditions, semiconcavity fail with p a point.)

The above geometrical problem can be restated as the minimum time probelm for the control system with set of velocities $V(x)$. Defining $F = \frac{Y+X}{2}$ and $G = \frac{Y-X}{2}$ this control system can be written in local coordinates in the form as:

$$\dot{x} = F(x) + uG(x), \quad F(x_0) = 0, \quad x \in M, \quad |u| \leq 1 \tag{5.1}$$

where F and G are $\mathcal{C}^\infty$ vector fields on M and now x_0 plays the role of p.

In the following we give the definition of topological Morse function and state our main result (Theorem 24). We call respectively *reachable set within time* $T > 0$ and *reachable set* the following sets:

$$\begin{aligned}
\mathcal{R}(T) \;&:=\; \{x \in M : \text{ there exists } t \in [0,T] \text{ and a trajectory} \\
&\qquad \gamma : [0,t] \to M \text{ of (5.1) such that } \gamma(0) = x_0,\ \gamma(t) = x\}. \qquad (5.2)\\
\mathcal{R} \;&:=\; \{x \in M : \text{ there exists } t > 0 \text{ and a trajectory} \\
&\qquad \gamma : [0,t] \to M \text{ of (5.1) such that } \gamma(0) = x_0,\ \gamma(t) = x\}.
\end{aligned}$$

The minimum time function $\mathbf{T}(\ .\ ) : \mathcal{R} \to \mathbb{R}^+$ (now we drop the label x_0) is by definition:

$$\mathbf{T}(x) := \min\{t \geq 0 : \text{ there exists a trajectory } \gamma(.) \text{ of (5.1) s.t. } \gamma(0) = x_0, \ \ \gamma(t) = x\} \tag{5.3}$$

Definition 30 *We say that a set $Q \subset M$ is thin if $Q = \cup_{i=1}^{n} M_i$, with $n \in \mathbb{N} \cup \{\infty\}$, $\{M_i\}_i$ locally finite, M_i connected embedded smooth one dimensional manifold with boundary. We say that a function $f : A \subset M \to \mathbb{R}$, A open, is piecewise $\mathcal{C}^2$ if it is $\mathcal{C}^2$ except a thin set Q. A smooth function is said Morse function if at critical points its Hessian is non singular. We say that a piecewise $\mathcal{C}^2$ function $f : A \subset M \to \mathbb{R}$ is topologically a Morse function if, taking Q minimal, the following holds. On the set $A \setminus Q$, f is a Morse function and for each $x \in Q$ there exist a neighborhood $I(x)$ of x and an homeomorphism $\psi : I(x) \to I(x)$ such that $f \cdot \psi : I(x) \to \mathbb{R}$ is a Morse function.*

Theorem 24 (**Morse Property**) *Fix $T > 0$. Under generic conditions on F and G, the minimum time function $\mathbf{T}|_{Int(\mathcal{R}(T))} : Int(\mathcal{R}(T)) \to \mathbb{R}^+$ is topologically a Morse function (here Int means topological interior).*

In Section 5.2 we define some basic tools and we states some results. In Section 5.3 we introduce the definition of *optimal strip*, similar to the one of *strip* used in Chapter 2, and we study the local properties of the MTF. In Section 5.4 we prove that the minimum time function is topologically a Morse function. In Section 5.5 we discuss the case of the two-dimensional initial manifold on the plane.

5.2 Basic Tools

Since in this chapter we deal with optimal trajectories we need the restriction of the trajectories $\gamma^{\pm}$. If $t_o^{\pm}$ are the last times in which $\gamma^{\pm}$ are optimal, we have $t_o^{\pm} \leq t_f^{\pm}$. We define $\gamma_o^{\pm} = \gamma^{\pm}|_{[0,t_o^{\pm}]}$.

In [31] it was shown that, under generic conditions, the problem of reaching in minimum time every point of the reachable set for the system (5.1) admits a regular synthesis. Moreover it was shown that $\mathcal{R}$ is a stratified subset of M (see Chapter 1 for the precise definition) such that the optimal synthesis can be obtained from a feedback $u(x)$ satisfying:

- on the regions of dimension 2, we have $u(x) = \pm 1$,
- on the regions of dimension 1, called frame curves (in the following FCs), $u(x) = \pm 1$ or $u(x) = \varphi(x)$.

The submanifolds of dimension 0 are called frame points (in the following FPs). Every FP is an intersection of two FCs. A FP x that is the intersection of F_1 and F_2 is called a (F_1, F_2) Frame Point. In [32] it was provided a complete classification of all types of FPs and FCs under generic conditions. All the possible FCs are the following (see also Chapter 2):

- FCs of Kind X (resp. Y), that correspond to subsets of γ_o^+ (resp. subsets of γ_o^-),

- FCs of kind C, called *switching curves*, that are curves made of switching points,
- FCs of kind S, i.e. turnpikes,
- FCs of kind K, called overlaps, reached optimally by two trajectories coming from different directions.
- FCs that are arcs of optimal trajectories that start at FPs. These trajectories "transport" special information. Here we need to distinguish three kind of these curves:
  - curves of kind γ_A that are abnormal extremals,
  - curves of kind γ_k that are curves starting at the terminal point of an overlap (i.e. that start at the FPs of kind $(Y,K)_3$ $(S,K)_2, (C,K)_1$, see below)
  - curves of kind γ_0 that are the other arcs of optimal trajectories that starts in FPs.

All the possible FPs are classified in [32]. There are seventeen topological equivalence classes of FPs: (X,Y), $(Y,C)_{1,2,3}$, (Y,S), $(Y,K)_{1,2,3}$, $(C,C)_{1,2}$, $(C,S)_{1,2}$, $(C,K)_{1,2}$, $(S,K)_{1,2}$, (K,K). The optimal synthesis near each Frame Point is showed in Figure 5.1.

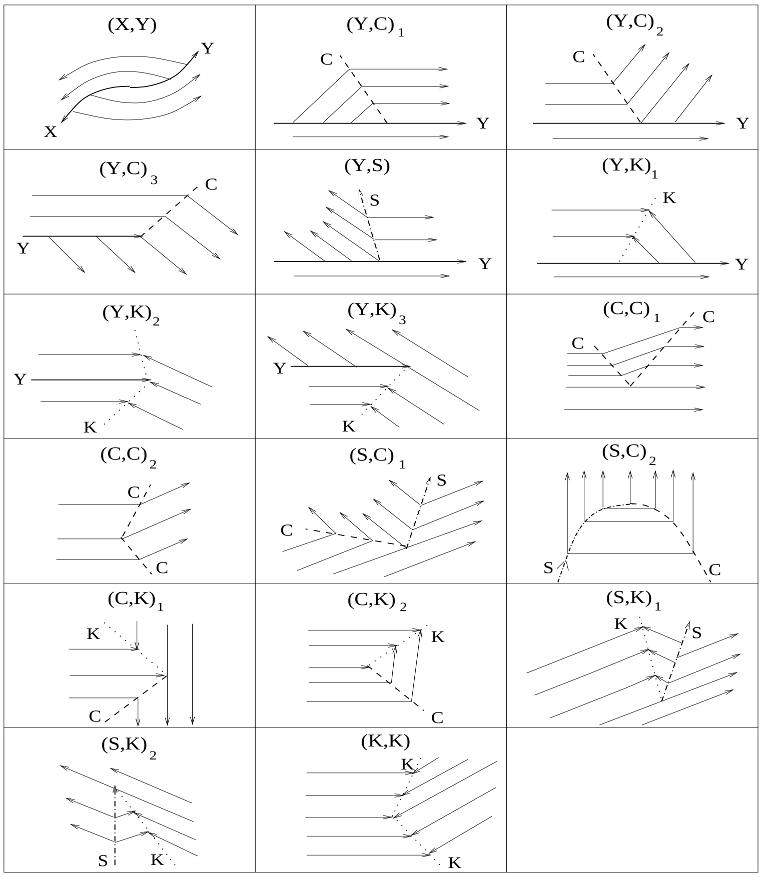


Figure 5.1: the FPs of the Optimal Synthesis.

Definition 31 *We define the following sets of bad points:*

- $bad_k := \{x \in \mathcal{R}$: *there exist two optimal trajectories* $\gamma_1,\ \gamma_2 : [0,T] \to M,\ T > 0$ *and* $\varepsilon > 0$ *such that* $\gamma_1(0) = \gamma_2(0) = x_0,\ \gamma_1(T) = \gamma_2(T) = x,\ \gamma_1(t) \neq \gamma_2(t)$ *for every* $t \in [T-\varepsilon, T[\}$.

- $bad_\gamma := \{x \in \mathcal{R}$: *there exist two optimal trajectories* $\gamma_1,\ \gamma_2 : [0,T] \to M,\ T > 0$, $a \in]0,T[$ *and* $\varepsilon > 0$ *such that* $\gamma_1(0) = \gamma_2(0) = x_0,\ \gamma_1(T) = \gamma_2(T) = x,\ \gamma_1(t) = \gamma_2(t)$ *for* $t \in [a,T]$ *and* $\gamma_1(t) \neq \gamma_2(t)$ *for* $t \in [a-\varepsilon, a[\}$.

- $bad := bad_k \cup bad_\gamma$.

Of course bad_k is the set of all the overlap FCs and bad_γ is the set of γ_k's. The minimum time front in time $T > 0$ (in the following MTF) is the T–level surface of $\mathbf{T}(x)$:

$$F_T = \{x \in \mathcal{R} : \quad \mathbf{T}(x) = T\}. \tag{5.4}$$

We want to prove that, under generic conditions on (F,G) and for generic $T > 0$ the corresponding MTF is a one dimensional piecewise $\mathcal{C}^1$ compact manifold. More precisely the result is valid for every T such that the following conditions holds.

(C1) No FP of kind $(Y,K)_2$, $(Y,K)_3$ or (K,K) is on the front F_T ,

(C2) The front F_T is not tangent to bad_k.

Definition 32 *Let* $x \in F_T$. *We define the contingent cone to the set* $\mathcal{R}$ *at* x *as:*

$$\begin{aligned} C^C(\mathcal{R}(\mathbf{T}(x)), x) \;\; = \;\; & \{v : \exists\ \{x_k\}_{k\in\mathbb{N}},\ \{h_k\}_{k\in\mathbb{N}},\ x_k \in \mathcal{R}(\mathbf{T}(x)), h_k > 0,\ h_k \to 0,\ \textit{such that} \\ & x_k = x + h_k v + o(h_k)\} \end{aligned}$$

and the normal cone as:

$$C^N(\mathcal{R}(\mathbf{T}(x)), x) = \{w : w \cdot v \leq 0,\ \textit{for all}\ v \in C^C(\mathcal{R}(\mathbf{T}(x)), x)\}$$

We have the following:

Theorem 25 (Smoothness Properties of F_T) *Under generic conditions on* (F,G), *for* $T > 0$ *satisfying conditions* **(C1)** *and* **(C2)**, F_T *is a one–dimensional piecewise* $\mathcal{C}^1$ *compact embedded submanifold of* M. *Moreover,* $x \in F_T$ *is a point in which* F_T *is not* $\mathcal{C}^1$ *iff:*

(case 1) $x \in bad$ *or*

(case 2) $x \in Supp(\gamma_o^+) \cup Supp(\gamma_o^-) \setminus (\gamma^+(t_f^+) \cup \gamma^-(t_f^-))$;

For every such x we have $C^N(\mathcal{R}(T), x) = \{0\}$ (case 1) or $Int(C^N(\mathcal{R}(T), x)) \neq \emptyset$ (case 2).

Theorem 26 (F_T on Abnormal Extremals) *Fix $T > 0$. Let $\bar{x} \in F_T$ and suppose $\bar{x} \notin bad$, $x \notin Supp(\gamma_o^+) \cup Supp(\gamma_o^-) \setminus (\gamma^+(t_f^+) \cup \gamma^-(t_f^-))$. Let $\bar{\gamma}$ be the unique optimal trajectory reaching $\bar{x}$. Then, under generic conditions, F_T is tangent to $Supp(\bar{\gamma})$ in $\bar{x}$ iff $\bar{\gamma}$ is an abnormal extremal.*

Moreover, from the analysis of [19, 32], it easily follows:

Theorem 27 (Topological Properties of of $\mathcal{R}(T)$) *For $T > 0$, $\mathcal{R}(T)$ is connected. Moreover if the synthesis does not contain any K FC, $\mathcal{R}(T)$ is simply connected. At the times in which* **(C1)** *or* **(C2)** *fails, the topology of the reachable set changes. In the planar case, there exists $n \in \mathbb{N}$ such that for t in a left neighborhood of T the reachable set $\mathcal{R}(t)$ is homeomorphic to a disc with n holes and for t in a right neighborhood of T, $\mathcal{R}(t)$ is homeomorphic to a disc with $n+1$ holes or $n-1$ holes. For a general manifold it may happen that we add or subtract a handle.*

Notice that if the first part of Theorem 27 applies, then the minimum time function has no critical point except x_0 and thus the level sets are homeomorphic to circles. More precisely, outside $Supp(\gamma_o^{\pm})$ the level sets are diffeomorphic to circles. In this case, trivially one concludes that the minimum time function topologically is a Morse function.

5.3 Properties of the Minimum Time Front

To prove Theorems 25 and 26, we study the smoothness properties of the MTF in a neighborhood of $Supp(\gamma_o^+) \cup Supp(\gamma_o^-)$ (see subsection 5.3.2 and 5.3.3), on the other bases of the strips (i.e the turnpikes, subsection 5.3.4), on the internal of the strips (subsection 5.3.5), on the overlap curves (subsection 5.3.6), and finally on the borders (subsection 5.3.7).

From now on, for simplicity, we consider the case $M = \mathbb{R}^2$ and $x_0 = 0$. The conclusions are valid in the general case mutatis mutandis.

5.3.1 Definition of Strips

Now we give the key definition of *optimal strip*. An optimal strip is essentially a one parameter continuous family of optimal bang–bang trajectories having the same switching strategy.

Let $c \in \mathbb{R}$, $d \in \mathbb{R} \cup \{+\infty\}$, $0 \leq c < d$. We set:

$$I_{c,d} := \begin{cases} [c, d] \text{ if } d \in \mathbb{R} \\ [c, d[\text{ if } d = \infty. \end{cases}$$

Definition 33 *Let a, b be two real numbers s.t. $0 \leq a < b$ (b possibly $+\infty$), $x \in \mathcal{R}$ and $f : I_{a,b} \to [a, \infty]$ a function such that $f(\alpha) \geq \alpha$. A set of trajectories $\mathcal{S}^{a,b,x,f} = \{\gamma_\alpha : I_{0,f(\alpha)} \to M,\ \alpha \in I_{a,b},\ \gamma_a(a) = x\}$ is called a strip if:*

i) $\forall\ \alpha \in I_{a,b},\quad \gamma_\alpha : I_{0,f(\alpha)} \to M$, *is an optimal trajectory for the control problem (5.1). Moreover there exists $\varepsilon > 0$ such that $\gamma_{[\alpha,\alpha+\varepsilon]}$ corresponds to a constant control ± 1;*

ii) $\forall\ \alpha \in]a, b[,\quad \gamma_\alpha$ *does not switch on $\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$ after time α;*

iii) *The set $\mathcal{B}^{a,b,x,f} = \{y \in \mathcal{R} :\ \exists\ \alpha \in\ ]a, b[$ and $t \in]\alpha, f(\alpha)[$ such that $y = \gamma_\alpha(t)$, t is a switching time for $\gamma_\alpha\}$ is never tangent to X or Y.*

iv) *The map $\eta :\ \alpha \in\ I_{a,b} \mapsto \gamma_\alpha(\alpha) \in M$ is a bang or singular arc.*

The function η is called the *base* of the strip, moreover $Int(\mathcal{S}^{a,b,x,f}) := \{\gamma_\alpha :\quad \alpha \in]a, b[\}$ is called an *open optimal strip* and $\partial\mathcal{S}^{a,b,x,f} := \{\gamma_a,\ \gamma_b\}$ is called *optimal strip border*.

The construction of the optimal synthesis,as in [17], can be done in the following way. One first constructs all optimal strips bifurcating from $\gamma_o^{\pm}$. Then one studies the evolution of each strip. It can happen that an optimal strip is divided into two strips when some trajectory of the strip enters a turnpike. This happens at Frame Points of kind $(S, C)_1$ and $(S, K)_1$ (moreover two strips with the turnpike as base start at these points). A strip can terminate on some K curve. Finally, a strip can glue together with another strip at Frame Points of kind $(Y, K)_3$, $(C, K)_1$ and $(S, K)_2$ (indeed the strip is divided into two parts one of which ends on the K curve and the other glues together with another strip). Let $\{S_M^i\}_{i \in I}$ be the set of all the *maximal strips* i.e. the strips with maximal base and maximal time. Clearly we may split the synthesis Γ as:

$$\Gamma = \bigcup_{i \in I} S_M^i.$$

This partition of the optimal synthesis can be realized with an algorithm enterely similar to the one of Chapter 2 and permits to study the evolution of the MTF and the properties of the minimum time function separately for the base, the borders, the internal of the strips and for the overlap trajectories. Notice that if $S^{a,b,x,f}$ is a maximal strip then $f(\alpha) < \infty$ for some $\alpha \in]a, b[$ iff there exists an overlap curve K such that $\gamma_\alpha(f(\alpha)) \in K$ for every $\alpha \in]a, b[$. Moreover notice that every FCs of the optimal synthesis that is not an overlap curve is the support of a base or the support of a border of an optimal strip.

5.3.2 F_T in a Neighborhood of x_0

Choose a system of coordinates such that:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(x) = \begin{pmatrix} -1 + O(x_1, x_2) \\ -b\, x_1 + b' x_2 + O(|x|^2) \end{pmatrix}, \quad b < 0. \tag{5.5}$$

The expressions for γ_o^+ and γ_o^- are:

$$\gamma_o^+(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \gamma_o^-(t) = \begin{pmatrix} -t + O(t^2) \\ \frac{1}{2} b t^2 + O(t^3) \end{pmatrix}. \tag{5.6}$$

Let us compute the MTF along γ_o^+, in the region where $x_2 < 0$. Consider a trajectory γ^α corresponding to constant control -1 on the interval $[0, \alpha[$ and to constant control +1 on the interval $[\alpha, T]$ where α belongs to the interval $[0, T]$. Using the expressions (5.6) it follows that $\gamma^\alpha(T)$ has coordinates $x_1(\alpha) = T - 2\alpha + O(\alpha^2)$, $x_2(\alpha) = \frac{1}{2} b \alpha^2 + O(\alpha^3)$, from which it follows the expression for the MTF:

$$x_2 = \frac{1}{8} b (x_1 - T)^2 + O(x_1^3). \tag{5.7}$$

Let as compute the MTF along γ_o^+, in the region where $x_2 > 0$. A point with $x_2 > 0$ is reached at time T by a trajectory corresponding to constant control +1 in the interval $[0, T - \alpha[$ and to constant control -1 in the interval $[T - \alpha, T]$ where α belong to the interval $[0, T]$. Consider now an optimal trajectory γ_η^- of (5.1), corresponding to constant control -1, and having initial condition $\gamma_\eta^-(0) = (\eta, 0)$. We have:

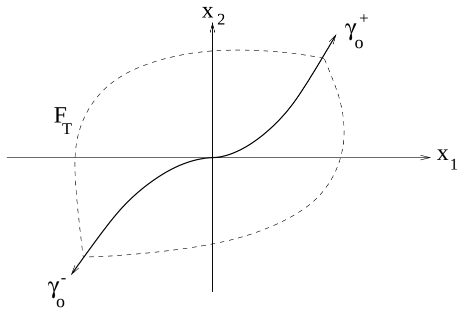
$$\gamma_\eta^-(t) = \begin{pmatrix} \eta - t + O(t^2, t\eta) \\ \frac{1}{2} b t^2 - b \eta t + O(t^3, \eta^2 t, \eta t^2) \end{pmatrix}. \tag{5.8}$$

Similarly to the previous case we obtain the expression for the MTF:

$$x_2 = \frac{1}{2} b T (x_1 - T) + \frac{3}{8} b (x_1 - T)^2 + O((x_1 - T)^2).$$

In the same way we may compute the MTF along γ_o^-, moreover the MTF is clearly smooth on $I(0) \setminus Supp(\gamma_o^+) \cup Supp(\gamma_o^-)$, where $I(0)$ is a neighborhood of the origin. This results can be summarized in the following (see figure):

Proposition 15 *For $T > 0$, sufficiently small, F_T is a 1–dimensional piecewise $\mathcal{C}^1$ embedded compact submanifold of M. The only two points in which F_T is not $\mathcal{C}^1$ are the points $F_T \cap Supp(\gamma_o^+)$ and $F_T \cap Supp(\gamma_o^-)$. Moreover if x is a such point we have $Int(C^N(\mathcal{R}(T), x)) \neq 0$.*



5.3.3 F_T in a Neighborhood of $\gamma_o^+ \cup \gamma_o^-$

Let $I(\gamma_o^{\pm})$ be a neighborhood of $Supp(\gamma_o^+) \cup Supp(\gamma_o^-)$. In this subsection we study the smoothness properties of $F_T \cap I(\gamma_o^{\pm})$. For this purpose we analyze F_T in a neighborhood of all the $(Y,\ .\ )$ FPs ordered by increasing time.

Analysis up to the first (YS) or $(YC)_1$ FP.
Assume $t_1'^+ > 0$ (being the case $t_1^+ > 0$ entirely similar) and suppose that at the point $\gamma_o^+(t_1'^+)$ a turnpike starts. In this case it bifurcates on the left of γ_o^+. Choose a system of coordinates such that:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(x) = \begin{pmatrix} a_0 + a_1(x_1 - t_1'^+) + a_2x_2 + O((x_1 - t_1'^+)^2, x_2^2) \\ b_0 + b_1(x_1 - t_1'^+) + b_2x_2 + O((x_1 - t_1'^+)^2, x_2^2) \end{pmatrix}, \tag{5.9}$$

where $a_0 < 1$, $b_0 > 0$. To have a (Y, S) point at $(t_0', 0)$ the following conditions must hold:

1) $(t_1'^+, 0) \in \Delta_B^{-1}(0)$ that implies:

$$a_2 = \frac{a_1 b_0}{a_0 - 1}.$$

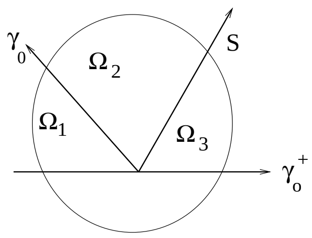
Under this condition, the set $\Delta_B^{-1}(0)$ (in I) is locally described by the equation:

$$x_2 = m_s(x_1 - t_1'^+) + O((x_1 - t_1'^+)^2) \text{ where } m_s = \frac{a_1a_2 - a_1b_1}{a_2^2 - a_1b_2},$$

where we assume the generic conditions $a_2^2 - a_1b_2 \neq 0$, $m_s \neq 0$. The case in which $a_2^2 - a_1b_2 = 0$ is the case in which the tangent to the turnpike is vertical.

2) X and Y point on opposite side of $\Delta_B^{-1}(0)$ that implies $arctan(b_0/a_0) > arctan(m_s)$ where we assume $[0, \pi[$ as co-domain of the function $arctan$.

Let us compute the evolution of the MTF in a neighborhood I of $(t_1'^+, 0)$. We refer to the following figure:



We have to compute the front F_T in the region $\Omega_1 \subset I$ enclosed by γ_o^+ (before time $t_1'^+$) and the trajectory γ_0 corresponding to constant control -1 starting from $(t_1'^+, 0)$, on the region Ω_2 enclosed by γ_0 and the turnpike, finally on the region Ω_3 between the turnpike and γ_o^+ (after time $t_1'^+$).

In Ω_1 using the expression of $X(x)$ given by formula (5.9), we obtain for F_T the expression:

$$x_2 = \frac{b_0}{a_0 - 1}(x_1 - T) + O((x_1 - t_1'^+)^2). \tag{5.10}$$

To compute the locus F_T in Ω_2 and Ω_3 we need the explicit expression for the singular trajectory. The velocity of the singular trajectory at a point $x \in S$ is the convex combination of $X(x)$ and $Y(x)$ corresponding to the direction of the turnpike. This guarantee the smoothness of the MTF on S (cfr also subsection 5.3.4) and gives us the expression for the singular trajectory γ^s ($S = Supp(\gamma^s)$):

$$\gamma^s(t) = \begin{pmatrix} t_1'^+ + v_1 t + O(t^2) \\ v_2 t + O(t^2) \end{pmatrix} \quad \text{where:}$$

$$v_1 = \frac{b_0 \left(a_1 {b_0}^2 - (-1+a_0)^2 b_2\right)}{a_1 b_0 \left(-1 + 2a_0 - {a_0}^2 + {b_0}^2\right) + (-1+a_0)^2 \left((-1+a_0)\, b_1 - b_0 b_2\right)} \tag{5.11}$$

$$v_2 = \frac{(-1+a_0)\, b_0\, (a_1 b_0 + b_1 - a_0 b_1)}{a_1 b_0 \left(-1 + 2a_0 - {a_0}^2 + {b_0}^2\right) + (-1+a_0)^2 \left((-1+a_0)\, b_1 - b_0 b_2\right)} \tag{5.12}$$

With similar arguments to the ones of the previous subsection we obtain the expression for F_T in Ω_2:

$$x_2 = v_2(T - t_1'^+) + \frac{(x_1 - t_1'^+) - v1(T - t_1'^+)}{(a_0 - v_1)}(b_0 - v_2) + O((x_1 - t_1'^+)^2).$$

Using (5.11) and (5.12), we obtain exactly the expression (5.10). This means that F_T is $\mathcal{C}^1$ at the intersection point with $Supp(\gamma_0)$. Exactly the same result holds in the case in which condition **2)** fails that is the case in which at $(t_1'^+, 0)$ a switching curve bifurcates from γ_o^+. In this case we have to use directly the PMP, but we omit this proof because it is entirely similar to the proof of Proposition 18 below. The smoothness properties of F_T in a neighborhood of $Supp(\gamma_o^+|_{]0,t_1'^+[})$ comes immediately from the shape of the synthesis in this region. All these results are summarized in the following:

Proposition 16 *Suppose $t_1'^+ > 0$, fix $y_0 \in Supp(\gamma_o^+|_{]0,t_1'^+]})$ and let t_0 be such that $y_0 = \gamma_o^+(t_0)$. There exists a ball B with center in y_0 such that if $\varepsilon > 0$ satisfies $\gamma_o^+(T) \in B$ for every $T \in]t_0 - \varepsilon, t_0 + \varepsilon[$, then $F_T \cap B$ is a one dimensional piecewise $\mathcal{C}^1$ embedded submanifold of M. Moreover F_T is not $\mathcal{C}^1$ only at the point $x = F_T \cap Supp(\gamma_o^+)$ and we have $Int(C^N(\mathcal{R}(T), x)) \neq 0$.*

Analysis up to the First $(Y,K)_1$ or $(Y,C)_2$ FP.

Assume $s_1^+ > 0$ (being the case $s_1'^+ > 0$ entirely similar). In this case a NTAE γ_A bifurcates on the right of $\gamma_o^+(s_1^+)$ and we can have two kind of FPs:

- $(Y,K)_1$ if an overlap curve starts at $\gamma_o^+(s_1^+)$,
- $(Y,C)_2$ if a switching curve starts at $\gamma_o^+(s_1^+)$.

The analysis of the first case is postponed to subsection 5.3.6 (cfr. also subsection 5.3.9). Let us consider the second case. Set $x = (x_1, x_2)$, and choose a local system of coordinates in such a way:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(x) = \begin{pmatrix} -1 & +O(x_1, x_2) \\ 0 & -bx_1 + b'x_2 + O(x^2) \end{pmatrix}, \tag{5.13}$$

$$X(x) = \begin{pmatrix} a & +O(x_1 - s_1^+, x_2) \\ 0 & +c(x_1 - s_1^+) + c'x_2 + O\left((x - (s_1^+, 0))^2\right) \end{pmatrix}, \tag{5.14}$$

where $b < 0$, $a \in]0,1[$, $c < 0$. The condition that at $(s_1^+, 0)$ a C curve start is (see [17]):

$$\delta := ac - b\left(\frac{1-a}{2}\right)^2 < 0, \tag{5.15}$$

and the expression for the switching curve was obtained in [17]:

$$x_2 = \frac{2c^2}{b(1-a)^2}(x_1 - s_1^+)^2 + O\left((x_1 - s_1^+)^3\right). \tag{5.16}$$

The expression for the NTAE γ_A can be obtained from (5.13):

$$\gamma_A(t) = \begin{pmatrix} x_1(t) = s_1^+ + at + O(t^2) \\ x_2(t) = c(\frac{1}{2}at^2) + O(t^3) \end{pmatrix}. \tag{5.17}$$

Thus we may describe $Supp(\gamma_A)$ as:

$$x_2^{\gamma_A}(x_1) = \frac{1}{2}\frac{c}{a}(x_1 - s_1^+)^2 + O((x_1 - s_1^+)^3). \tag{5.18}$$

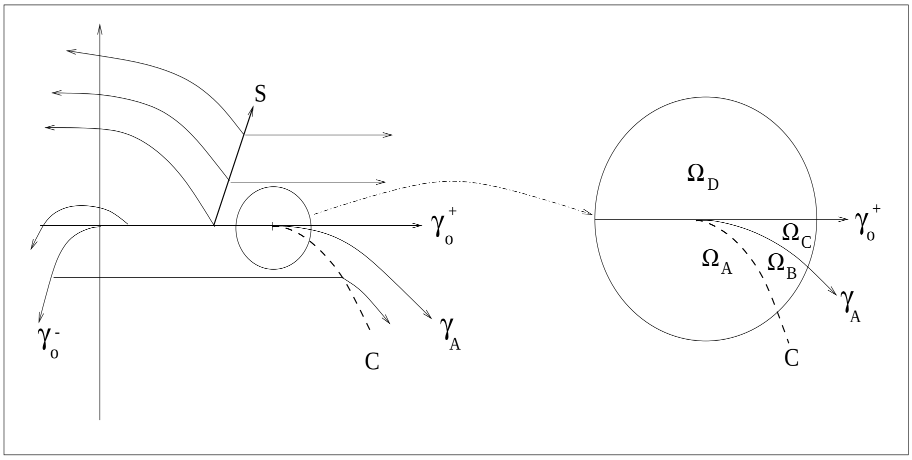


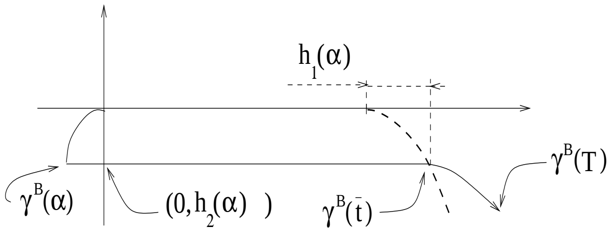
Figure 5.2

Divide I in the four region Ω_A, Ω_B, Ω_C, Ω_D delimited by γ_o^+, C and γ_A as in figure 5.2. In the following we compute F_T in these regions and we check that on C and γ_A the tangents to F_T from both sides coincide.

In Ω_A the expression for F_T is obtained similarly to (5.7):

$$x_2^A(x_1) = \frac{1}{8}b(x_1 - T)^2 + O(x_1^3). \tag{5.19}$$

A point $x^B \in \Omega_B$ is reached by an optimal trajectory γ^B corresponding to constant control -1 in the interval $[0, \alpha]$, to constant control +1 in the interval $[\alpha, \bar{t}]$ (where $\bar{t}$ is the time in which this trajectory intersects the C curve) and to constant control -1 in the interval $[\bar{t}, T]$.



Let us parameterize the curve C with α. Using (5.16) we obtain the expression:

$$\begin{cases} x_1(\alpha) = s_1^+ + h_1(\alpha) + O(\alpha^2) \\ x_2(\alpha) = h_2(\alpha) + O(\alpha^3), \end{cases}$$

$$\text{where } h_1(\alpha) := \frac{b(1-a)}{2c}\alpha, \qquad h_2(\alpha) := \frac{1}{2}b\alpha^2.$$

The expression for an optimal trajectory γ^h of (5.1) corresponding to constant control -1, and having initial condition $\gamma^h(0) = (s_1^+ + h_1, h_2) \in I(0)$ is:

$$\gamma^h(t) = \begin{pmatrix} x_1(t) = s_1^+ + h_1 + at + O(t^2, h_1 t, h_2 t) \\ x_2(t) = h_2 + c(\frac{1}{2}at^2 + h_1 t) + O(t^3, h_1 t^2, h_1^2 t, h_2 t^2, h_2^2 t) \end{pmatrix} \tag{5.20}$$

It follows that $x^B \in F_T \cap \Omega_B$ has coordinates:

$$\begin{cases} x_1^B(\alpha) = s_1^+ + h_1(\alpha) + a\beta(\alpha) + O(\alpha^2) \\ x_2^B(\alpha) = h_2(\alpha) + c(\frac{1}{2}a\beta(\alpha)^2 + h_1(\alpha)\beta(\alpha)) + O(\alpha^3), \end{cases}$$

where $\beta(\alpha) := T - \bar{t} = T - h_1(\alpha) - s_1^+ - 2\alpha$. Hence the expression for F_T in Ω_B:

$$\begin{aligned} x_2^B(x_1) &= \frac{1}{2}ac(T - s_1^+)^2 + c(T - s_1^+)x_1' + c\frac{(2-a)b + 4c}{8\delta}x_1'^2 + O(x_1'^3) \\ &\quad \text{where } x_1' := (x_1 - s_1^+ - a(T - s_1^+)). \end{aligned} \tag{5.21}$$

Finally, the front $F_T \cap \Omega_C$ can be obtained by lengthy but straightforward computations:

$$\begin{aligned} x_2^C(x_1) &= c\frac{x_1'}{1-a}(T - s_1^+ - \frac{x_1'}{1-a}) + \frac{1}{2}ac(T - s_1^+ - \frac{x_1'}{1-a})^2 + \\ &\quad O((T - s_1^+ - \frac{x_1'}{1-a})^3) = \\ &= \frac{1}{2}ac(T - s_1^+)^2 - c(\frac{1}{2}a - 1)(T - s_1^+)\frac{x_1'}{1-a} + c(\frac{1}{2}a - 1)(\frac{x_1'}{1-a})^2 + \\ &\quad O((\frac{x_1'}{1-a})^3). \end{aligned} \tag{5.22}$$

Now it is easy to check the following:

- $\dot{x}_2^A(\tilde{x}_1) = \dot{x}_2^B(\tilde{x}_1)$ where $\tilde{x}_1$ is the intersection of the two fronts with the C curve. This means that F_T is $\mathcal{C}^1$ in $\tilde{x}_1$.
- $\dot{x}_2^B(\bar{x}_1) = \dot{x}_2^C(\bar{x}_1) = \dot{x}_2^{\gamma_A}(\bar{x}_1)$ where $\bar{x}_1$ is the intersection of the two fronts with $Supp(\gamma_0)$. This means that F_T is $\mathcal{C}^1$ in $\bar{x}_1$ and it is tangent to γ_A in $\bar{x}_1$.

Analysis of the other Frame Points.
The front F_T at the other singular points along $\gamma_o^\pm$ can be analyzed by arguments completely similar to those used above. The only delicate points are the ones in which $\gamma^\pm$ stops to be optimal (namely $\gamma^\pm(t_f^\pm)$). At this points we can have a $(Y,K)_3$ or a $(Y,C)_3$ FP. In the first case, that is the only case in which a γ_k trajectory generates from $\gamma_o^\pm$, it is easy to check that the MTF form a corner on γ_k. The second case, can be studied as the interior of strips (see below). We obtain that F_T is always a piecewise $\mathcal{C}^1$ manifold with corners only on $\gamma_o^\pm$. Note that a NTAE may generate from a $(Y,C)_3$ points and in this case we can prove that the front is tangent to the NTAE by arguments similar to those used above.

5.3.4 The MTF on the Other Bases

The front F_T on the singular trajectories is described by the following:

Proposition 17 *Suppose that there exists a point $y^0 \in F_T \cap S$ where S is some open turnpike of the optimal synthesis. Then F_T is $\mathcal{C}^1$ in y^0.*

Proof. It is a consequence of the fact that the velocity of the singular trajectory at a point $x \in S$ is the convex combination of $X(x)$ and $Y(x)$ corresponding to the direction of the turnpike. ■

5.3.5 The MTF on Open Strips

In this subsection we prove that the MTF restricted to an open strip is always $\mathcal{C}^1$.

Observation **5** Notice that the $\mathcal{C}^1$ property of the MTF can also be obtained reasoning on the position of the covectors with respect to the MTF. More precisely, each covector has negative scalar product with all variational vectors corresponding to needle variations. From the structure of the optimal synthesis it is clear that fixed an optimal trajectory the neighboring trajectories of the optimal synthesis can be obtained by needle variations. This implies that the MTF of a strip can not form angles in such a way that the reachable set is locally concave. On the other side, after the first switching the covector is determined up to the multiplication by a positive constant, hence also angles of the MTF that render the reachable set locally convex are prohibited.

Fix an optimal strip $\mathcal{S}^{a,b,x,f}$ and let $T \in]a, \sup\{f(\alpha), \;\; \alpha \in]a,b[\}[$. We define the strip-MTF (in the following s-MTF) to be the set:

$$F_T^{a,b,x,f} := \{x \in F_T : \;\; x = \gamma_\alpha(T) \text{ for some } \gamma_\alpha \in \mathcal{S}^{a,b,x,f}, \;\; \alpha \in]a,b[\}.$$

Proposition 18 *Let $\mathcal{S}^{a,b,x,f}$ be an optimal strip, $T \in]a, \sup\{f(\alpha), \;\; \alpha \in]a,b[\}[$ and F_T the corresponding s-MTF. Then $F_T^{a,b,x,f}$ is a one–dimensional $\mathcal{C}^1$ embedded submanifold of M. Moreover if $T \in]a, \inf\{f(\alpha), \;\; \alpha \in]a,b[\}[$ then $F_T^{a,b,x,f}$ is also connected.*

The proof follows from the next two Lemmas. With the first we prove that the s-MTF is smooth when it intersects the "first" switching locus. With the second we prove that if the s-MTF is $\mathcal{C}^1$ when it intersect a switching locus, then it is $\mathcal{C}^1$ when it intersects the subsequent. By induction we reach the conclusion.

Lemma 4 *Let $Int(\mathcal{S}^{a,b,x,f})$ be an open optimal strip. Fix $\alpha_0 \in]a,b[$ and suppose that $T_0 \in]\alpha_0, f(\alpha_0)[$ is the first switching time of γ_{α_0} after α_0. Then $F_{T_0}^{a,b,x,f}$ is $\mathcal{C}^1$ in $y^0 := \gamma_{\alpha_0}(T_0)$.*

Proof. Suppose that γ_{α_0} corresponds to constant control $+1$ in the interval $]\alpha_0, T_0[$. Choose a local system of coordinates in such a way that $Y = (1,0)$, $\gamma_{\alpha_0}(\alpha_0) = (0,0)$ and the expression for the base of the strip is:

$$\eta(t) = \begin{pmatrix} 0 \\ t - \alpha_0 \end{pmatrix}, \quad t \in]a,b[.$$

In this system of coordinates we have $y^0 = (T_0 - \alpha_0, 0)$ and the expression of G on the base is ($t \in]a, b[$):

$$G(\eta(t)) = \begin{cases} \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{in the case in which the base is a } X\text{–trajectory,} \\ \qquad \text{so } X(0,t) = (0,1) \\ \frac{1}{1-\varphi(\eta(t))}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{in the case in which the base is a } Z\text{–trajectory,} \\ \qquad \text{so } (F + \varphi G)(0,t) = (0,1). \end{cases} \tag{5.23}$$

Let λ_α be the covector associated to γ_α ($\alpha \in]a, b[$). From the condition $\lambda_\alpha(\alpha) \cdot G(\gamma_\alpha(\alpha)) = 0$ we obtain:

$$\lambda_\alpha(\alpha) = \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The switching curve C passing from y^0 is determined by the equation:

$$\lambda_\alpha(t) \cdot G(\gamma_\alpha(t)) = 0. \tag{5.24}$$

Let t_α be the first switching time of γ_α after α, then $\dot{\lambda}_\alpha(t) = 0$ for $t \in [\alpha, t_\alpha]$. We get that condition (5.24) is equivalent to:

$$G_1(x) + G_2(x) = 0, \tag{5.25}$$

where $G_1(x)$ and $G_2(x)$ are the two components of G. In terms of components of the vector field X, (5.25) can be written as:

$$X_1(x) + X_2(x) = 1. \tag{5.26}$$

We suppose that at y^0, $F_T^{a,b,x,f}$ and C are not tangent otherwise the conclusion follows easily. Assume that we are in the situation of the following figure (the other case being similar):

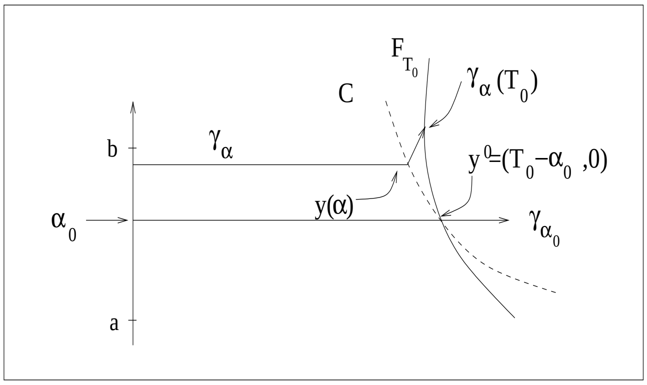


Figure 5.3

In a neighborhood of the point y^0 we may parameterize the C curve by a function:

$$y(\alpha) = \left(\begin{array}{c} T_0 - \alpha_0 - a(\alpha) \\ \alpha \end{array} \right), \tag{5.27}$$

where $a(.)$ is a $\mathcal{C}^1$ function defined in a neighborhood of $\alpha = 0$ and satisfying $a(0) = 0$. At time T_0, a trajectory with $\alpha > 0$ reaches the point:

$$z(\alpha) := \gamma_\alpha(T_0) = y(\alpha) + (a(\alpha) - \alpha)X(y^0) + O(\alpha^2). \tag{5.28}$$

It follows

$$\dot{z}(0) = \left(\begin{array}{c} -\dot{a}(0) + (\dot{a}(0) - 1)X_1(y^0) \\ 1 + (\dot{a}(0) - 1)X_2(y^0) \end{array} \right).$$

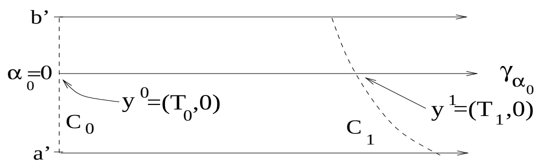
While in the region such that $x_2 < 0$ we have that the s–MTF has direction $(-1, 1)$. The front $F_T^{a,b,x,f}$ is $\mathcal{C}^1$ in y^0 iff $\dot{z}(0)$ is parallel to $(-1, 1)$. This condition is satisfied since (5.26) holds. ■

Lemma 5 *Let $Int(\mathcal{S}^{a,b,x,f})$ be an open optimal strip. Fix $\alpha_0 \in]a, b[$ and suppose that $T_0, T_1 \in]\alpha_0, f(\alpha_0)[$ are two subsequent switching times of γ_{α_0} after α_0. Suppose that $F_{T_0}^{a,b,x,f}$ is $\mathcal{C}^1$ in $y^0 := \gamma_{\alpha_0}(T_0)$. Then $F_{T_1}^{a,b,x,f}$ is $\mathcal{C}^1$ in $y^1 := \gamma_{\alpha_0}(T_1)$.*

Proof. Suppose that γ_{α_0} corresponds to constant control $+1$ in the interval $]T_0, T_1[$. Set $x = (x_1, x_2)$, choose a local system of coordinates in such a way $Y = (1, 0)$, $\gamma_{\alpha_0}(T_0) = (0, 0)$, the expression for the switching curve C_0 passing from y^0 is:

$$C_0 = \{x \in M : x_1 = 0, \quad x_2 \in]a', b'[\quad a' < 0, \quad b' > 0\},$$

and $X(0, x_2) = (\beta, \beta)$ ($\beta \in \mathbb{R} \setminus \{0\}$) for $x_2 \in]a', b'[$.



In this system of coordinates we have $y^1 = (T_1 - T_0, 0)$ and the expression of G on C_0 is:

$$G((0, x_2)) = \frac{1}{2} \left(\begin{array}{c} 1 - \beta \\ -\beta \end{array} \right) \text{ for every } x_2 \in]a', b'[. \tag{5.29}$$

Using the same reasoning as in the proof of the previous Lemma, we have that the switching curve C_1 passing through y^1 is determined by the condition:

$$X_1(x) + \frac{1 - \beta}{\beta} X_2(x) = 1. \tag{5.30}$$

Suppose that we are in the same situation of the proof of Lemma 4 (see figure 5.3). At time T_1 a trajectory with $\alpha > 0$ reaches the point:

$$z(\alpha) := \gamma_\alpha(T_1) = y(\alpha) + (\dot{T}(0)a(\alpha) - \alpha)X(y^1) + O(\alpha^2), \tag{5.31}$$

where:

$$y(\alpha) = \begin{pmatrix} T_1 - T_0 - a(\alpha) \\ \alpha \end{pmatrix},$$

(being $a(.)$ a $\mathcal{C}^1$ function defined in a neighborhood of $\alpha = 0$ and satisfying $a(0) = 0$) and $T(\alpha)$ is the switching time of γ_α on the switching curve passing from y^0. It follows:

$$\dot{z}(0) = \begin{pmatrix} -\dot{a}(0) + (\dot{T}(0)\dot{a}(0) - 1)X_1(y^1) \\ 1 + (\dot{T}(0)\dot{a}(0) - 1)X_2(y^1) \end{pmatrix}.$$

This vector must be parallel to the vector $(-\dot{T}(0), 1)$, i.e. the following conditions must hold:

(c1) $\dot{T}(0) = \frac{1-\beta}{\beta}$ if $\dot{T}(0) \neq 0$,

(c2) $\beta = 1$ if $\dot{T}(0) = 0$.

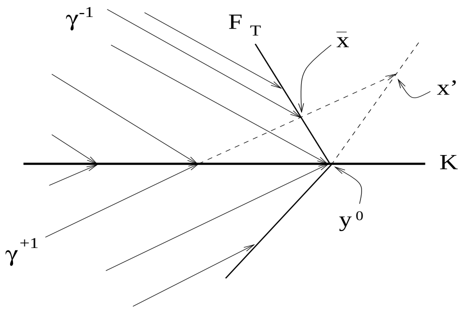
Imposing that $F_{T_0}^{a,b,x,f}$ is $\mathcal{C}^1$ in $y^0 := \gamma_{\alpha_0}(T_0)$, it is easy to check that **(c1)** and **(c2)** are verified. ■

5.3.6 F_T on the Overlap Curves

The MTF along overlap curves is described by the following:

Proposition 19 *Let $y^0 = F_T \cap K$ where K is some overlap curve of the optimal synthesis and suppose that condition* **C2)** *(at the beginning of Section 5.2) holds. Then there exists a ball B centered at y^0 such that $F_T \cap B$ is not $\mathcal{C}^1$ only at y^0. Moreover generically $C^N(\mathcal{R}(T), x) = \{0\}$.*

Proof. Choosing B sufficiently small, we have that y^0 is the only possible point in which F_T is not $\mathcal{C}^1$, indeed $(F_T \cap B) \setminus \{y^0\}$ is contained in interiors of strips. By condition **C2)**, the two connected components of $(F_T \cap B) \setminus \{y^0\}$ cannot be tangent to K. Moreover the K curve is reached from the two sides by trajectories with constant control $+1$ and -1 respectively. By contradiction suppose that we have $Int(C^N(\mathcal{R}(T), y^0)) \neq \emptyset$, that means that we have the situation of the following figure:



The point $\bar{x}$ is reached by a trajectory γ^{-1} (corresponding to constant control -1) at time T, while it is reached at time $T' < T$ by a trajectory γ^{+1} corresponding to constant control $+1$. This is due to the fact that γ^{+1} reaches x' at time T. This contradict the optimality of γ^{-1}. The situation in which $(F_T \cap B)$ is $\mathcal{C}^1$ in y^0 is not generic. It follows the conclusion. ■

5.3.7 F_T on the Borders

In this section we study the smoothness property of the MTF on strip borders. In the first and second subsections we analyze respectively the strip borders of kind γ_0. In the third subsection we analyze the strip borders of kind γ_k. Since at the two sides of a border of a strip the trajectories have the same control, we have to analyze only the FPs at which this border generates. The FPs of kind $(Y, .)$ are already studied in subsection 5.3.3. Here we complete the analysis.

F_T on Strip Border of Kind γ_0

The strip borders of kind γ_0, that do not bifurcate from $\gamma_o^{\pm}$, are generated at the following FPs: $(C,C)_1$, $(C,S)_{1,2}$, $(S,K)_1$, and at $(C,C)_2$ if $\Delta_A \neq 0$. We analyze in details the case $(C,S)_1$ and $(S,K)_1$.

Assume to have a $(C,S)_1$ or a $(S,K)_1$ FP at $y^1 := \gamma_0(t_1)$ and suppose the following:

- the previous switching time of γ_0 is $t_0 < t_1$;
- γ_0 corresponds to constant control $+1$ in the interval $[t_0, t_1]$;
- $y^0 := \gamma_0(t_0)$ belongs to a X FC (being similar the cases in which it belong to a switching curve or to a turnpike).

Set $T_0 = t_1 - t_0$. Choose a system of coordinates and shift the time in such a way $y^0 = \gamma_0(0) = 0$, and:

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X(0, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X(x) = \left(\begin{array}{l} a_0 + a_1(x_1 - T_0) + a_2 x_2 + O((x_1 - T_0)^2, x_2^2) \\ b_0 + b_1(x_1 - T_0) + b_2 x_2 + O((x_1 - T_0)^2, x_2^2) \end{array} \right), \quad a_0 < 1, \quad b_0 > 0.$$

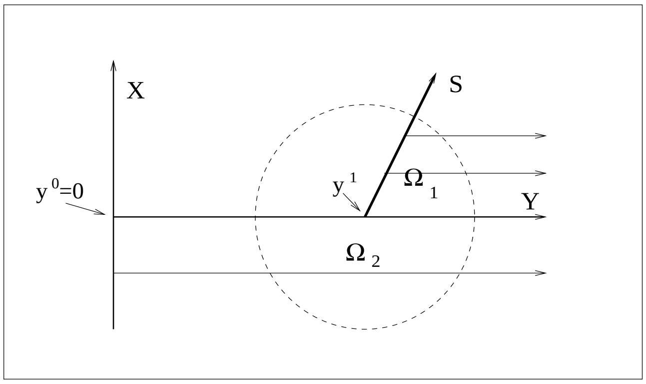


Figure 5.4

To have a $(C,S)_1$ or a $(S,K)_1$ FP at y^1 the following condition must hold (cfr. subsection 5.3.3):

$$a_2 = \frac{a_1 b_0}{a_0 - 1}. \tag{5.32}$$

Exactly as in subsection (5.3.3), we obtain the equation of the singular trajectory (defined in a right neighborhood of T_0):

$$\gamma^s(t) = \left(\begin{array}{l} T_0 + v_1 t + O(t^2) \\ v_2 t + O(t^2) \end{array} \right),$$

where v_1 and v_2 are given by formulas (5.11) and (5.12). Let I a neighborhood of y^1 and let us compute the MTF in the region Ω_1 enclosed by γ_0 (after T_0) and S (see figure 5.4). Consider a trajectory corresponding to constant control $+1$ in $[0, T_0[$, to the singular control $\varphi(\gamma^s(t))$ in $[T_0, T_0 + \alpha[$ and to constant control $+1$ in $[T_0 + \alpha, T]$. At time T this trajectory reaches the point:

$$z(\alpha) = (T_0, 0) + \alpha \dot{\gamma}^s(0) + (T - T_0 - \alpha) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) + O(\alpha^2) \in \Omega_1.$$

It follows:

$$\dot{z}(0) = \dot{\gamma}^s(0) - \left(\begin{array}{c} 1 \\ 0 \end{array} \right).$$

The MTF is $\mathcal{C}^1$ at $\gamma_0(T)$ iff $\dot{z}(0)$ is parallel to $(1, -1)$, so we have to check the condition:

$$v^1 + v^2 = 1. \tag{5.33}$$

Let $\lambda(.)$ be the covector associated to $\gamma_0(.)$. From the following equations:

$$\begin{aligned}
&\lambda(0) \cdot G(\gamma_0(0)) = 0, \\
&\dot{\lambda}(t) = 0 \ \ \forall\, t \in [0, T_0], \\
&\lambda(T_0) \cdot G(\gamma_0(T_0)) = 0, \\
&\lambda(T_0) \cdot [F, G](\gamma_0(T_0)) = 0,
\end{aligned}$$

we obtain:

$$a_0 + b_0 = 1, \qquad a_1 + b_1 = 0. \tag{5.34}$$

Using (5.34) and (5.32) one check immediately that condition (5.33) is verified. ■

F_T on Strip Border of Kind γ_A and Proof of Theorem 26

After the first switching, the only FP involving an abnormal extremal is the point $(C, C)_2$ where Δ_A vanishes. The analysis is completely similar to that of $(Y, C)_2$. The MTF is $\mathcal{C}^1$ on γ_A and, by induction, it is possible to prove that it is always tangent to $Supp(\gamma_A)$. This proves Theorem 26.

F_T on Strip Border of Kind γ_k

The strip borders of kind γ_k, that do not bifurcate from $\gamma_o^{\pm}$, are generated at the following FPs: $(C, K)_1, (S, K)_2$. Again it is easy to check that the MTF form a corner on γ_k. Moreover, for every $x \in \gamma_k$, one has $C^N(\mathcal{R}(\mathbf{T}(x)), x) = \{0\}$.

5.3.8 Proof of Theorem 25

The front at points in the interior of a strip or a strip border has been analyzed in the previous sections.

Assume that x is a Frame Point. If x is not of kind $(Y, K)_{2,3}$ or (K, K), that is if **C1)** does not hold, then an analysis completely similar to that of γ_0 strip borders permits to reach the conclusion.

Finally, if x is on a frame curve of kind K, that is where a strip may end, and the condition **C2)** does not hold then we can apply Proposition 19.

This concludes the proof of Theorem 25. ■

5.3.9 The Extremal Front

If we consider all extremal trajectories, including the not optimal ones, we can define the extremal front F_T^e as the set of points that are reached by an extremal trajectory at time

T. Obviously we have $F_T \subset F_T^e$. Our aim is to study the singularities developed by F^e as a two dimensional wave front.

It is well known that for two dimensional wave fronts there is only a generic kind of singularity called *standard singularity*, see [3, 4]. We want prove that at Frame Points of the optimal synthesis the wave front F^e develops only standard singularities.

We describe the situation only at the Frame Point $y^0 = \gamma_o^+(s_1^+)$ that is of kind $(Y,K)_1$ being the other cases $(C,K)_2$ and $(S,K)_1$ similar. The extremal synthesis, that is the collection of all extremals in the cotangent bundle (see Chapter 2 for details), at $(Y,K)_1$ has a cusp projection singularity. The borders of the cusp are the X trajectory from y^0 (that is a γ_0 Frame Curve for the extremal synthesis) and a curve of kind $\bar{C}$, that is a curve of switching points reached by not optimal extremal trajectories that reflects "back" after the switching.

Let $x = (x_1, x_2)$ and choose a system of coordinates in such a way (5.13) and (5.14) hold (but with the condition $a \in]0,1[$ replaced by $a < 1$). The fact that at $(s_1^+, 0)$ a K curve starts implies (see [17]):

$$\delta := ac - b\left(\frac{1-a}{2}\right)^2 > 0. \tag{5.35}$$

For time $t < \mathbf{T}(y^0)$, near $\mathbf{T}(y^0)$, $F_t^e = F_t$ and the front F_t^e is a one dimensional piecewise smooth manifold. For time $t > \mathbf{T}(y^0)$, near $\mathbf{T}(y^0)$, the front F_t^e is formed by three branches:

- the first corresponds to trajectories that bifurcate from γ^+ after x_0. Its expression as a function of x_1 is an arc of the function $x_2^A(x_1)$ defined in (5.19);
- the second corresponds to trajectories that switched on the $\bar{C}$ curve. Its expression as a function of x_1 is an arc of the function $x_2^B(x_1)$ defined in (5.21);
- the third corresponds to Y trajectories that did not yet reach the curve $\bar{C}$. Its expression as a function of x_1 is an arc of the function $x_2^C(x_1)$ defined in (5.22).

Let α^A, α^B, α^C be the coefficient of x_1^2 in the expression of x_2^A, x_2^B, x_2^C respectively. With our choice of parameters (i.e. $b,\ c < 0,\ a < 1,\ \delta > 0$), one can check that $\alpha^A < 0,\ \ \alpha^B > \alpha^C > 0$. Moreover $x^A(x_1),\ x^B(x_1)$ have a first order tangency at the point:

$$x^{AB} = \frac{(a-1)bt + 4cs_1^+}{(a-1)b + 4c},$$

while $x^B(x_1)$ and $x^C(x_1)$ have a first order tangency between them and with the trajectory exiting y^0 with constant control -1 at the point

$$x^{BC} = s_1^+ + a(t - s_1^+).$$

We have $x^{AB} > x^{BC}$ thus the front F^e develops a standard singularity.

Notice that all standard singularities of F^e are developed at Frame Points (namely $(Y,K)_1$, $(C,K)_2$ and $(S,K)_1$) corresponding to cusp projection singularities of the extremal synthesis (see Chapter 2). Figure 5.5 shows the extremal synthesis and F^e_t, near the point $(Y,K)_1$, in the two cases in which $a > 0$ and $a < 0$. The case $a = 0$ is not generic.

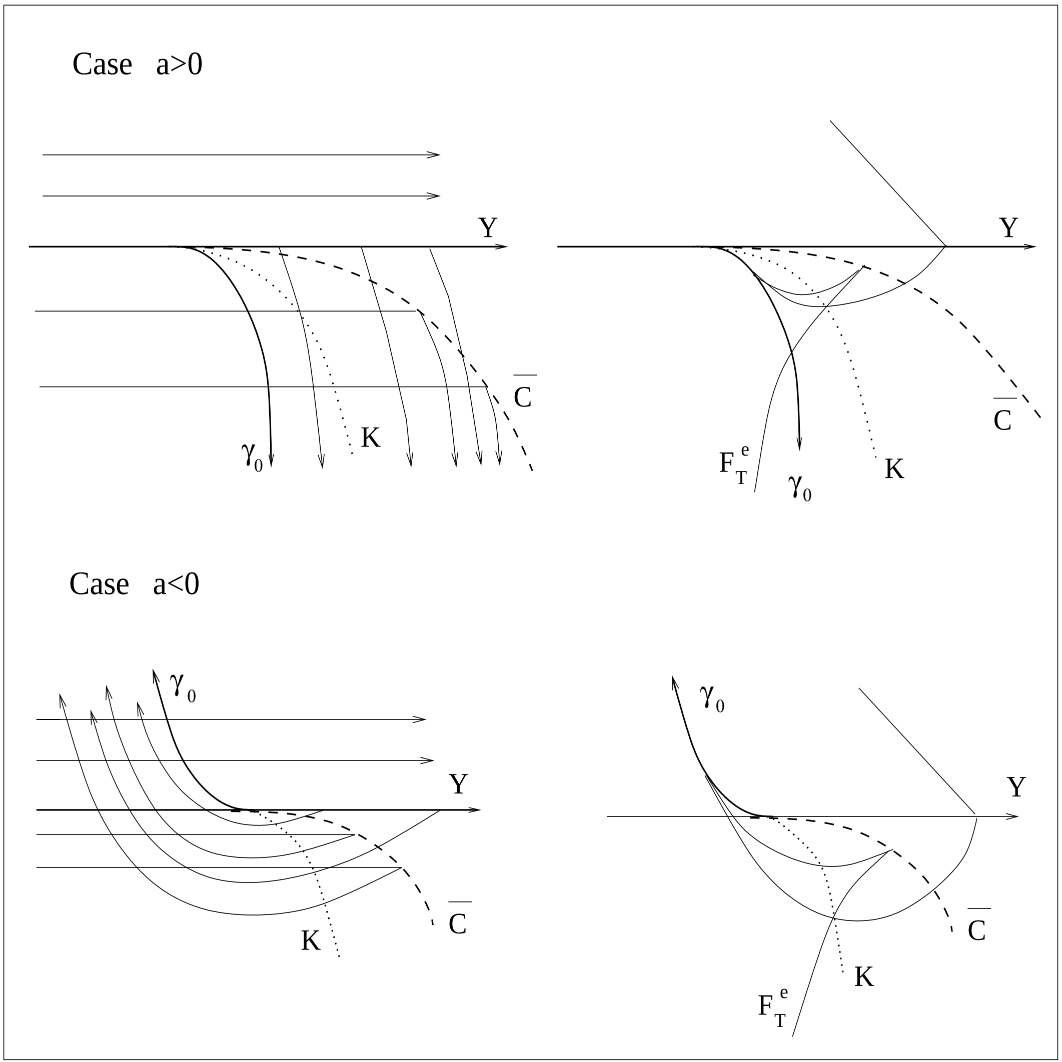


Figure 5.5: The extremal synthesis and the MTF in a neighborhood of the first FP of kind $(Y,K)_1$

5.4 Proof of Theorem 24

We are going to prove Theorem 24 analyzing the MTF first at points inside a strip, then at points on Frame Curves and finally at Frame Points.

If x is point inside a strip, not on a C curve, the wave fronts are locally smooth one dimensional manifolds. Moreover, the optimal synthesis is locally a *flow box* of one of the vector fields X and Y. It easily follows that the minimum time function $\mathbf{T}$ is smooth and x is not a critical point, since $\mathbf{T}$ increases in the direction of X or Y.

Assume now that x belongs to a Frame Curve (but is not a Frame Point). If the Frame Curve is of kind X, Y, γ_0 or S then the front is a piecewise $\mathcal{C}^1$ manifold and the minimum time is increasing along the Frame Curve, hence the function $\mathbf{T}$ is locally topologically equivalent to a nonzero linear function. Notice that indeed the front is $\mathcal{C}^1$ in case of γ_0 and S curves. If the Frame Curve is of kind C then the front is $\mathcal{C}^1$ and there is an increasing direction (indeed two directions X and Y) so $\mathbf{T}$ is $\mathcal{C}^1$ and topologically equivalent to a nonzero linear function. We are left with the case of K and γ_k Frame Curves. Given a Frame Curve D of K type, there exist two Frame Curves D_1 and D_2 such that the curve D is determined by the solution of the following system, see [17]:

$$\begin{cases} (exp\ tX)\,\alpha_1(s) \\ (exp\ t'Y)\,\alpha_2(s') \\ s+t=s'+t' \end{cases},$$

where $(exp\ tV)\bar{x}$ indicates the integral trajectory at time t of the vector field V starting from $\bar{x}$ and α_1 and α_2 are two smooth parameterizations of the curves D_1 and D_2 respectively. It is clear that under generic assumptions on X and Y, hence on F and G, the above system determine a smooth curve D such that the minimum time function $\mathbf{T}$ restricted to D is a Morse function. We have now three cases:

a) x is not critical for $\mathbf{T}$ on D,

b) x is a maximum for $\mathbf{T}$ on D,

c) x is a minimum for $\mathbf{T}$ on D.

In case a), we are in the situation of Proposition 19, the MTFs near x are piecewise $\mathcal{C}^1$ manifolds. The function $\mathbf{T}$ is increasing (or decreasing) in the direction tangent to K and is topologically equivalent to a nonzero linear function.

Assume case b) holds, then x is not a Frame Point and condition **C2)** is violated. Since $\mathbf{T}$ is increasing along the two strips ending at K, x is a maximum for $\mathbf{T}$. The MTF through x reduces to a point and is homeomorphic to a circle for points in a neighborhood of x. Therefore $\mathbf{T}$ is topologically equivalent to $f(x_1,x_2)=-x_1^2-x_2^2$ at the origin, that is x is topologically a regular maximum.

If c) holds, then either x is a Frame Point or x is not a Frame Point and condition **C2)** does not hold. Assume the latter holds. Again, $\mathbf{T}$ is increasing along the two strips ending at K. The MTF at x is formed by two tangent parabolas through x, for points in a neighborhood is formed by two branches that are smooth if they do not intersect the K

curve. Therefore $\mathbf{T}$ is topologically equivalent to $f(x_1, x_2) = x_1^2 - x_2^2$ at the origin, that is x is topologically a regular saddle.

If the Frame Curve is of kind γ_k then we are in case a) above, indeed $\mathbf{T}$ is increasing along γ_k, and we conclude in the same way.

We are left with the case of x Frame Point. The points along $\gamma^{\pm}$ have been analyzed above and the minimum time function $\mathbf{T}$ is topologically equivalent to a linear function at those points. The same happens for the Frame Points of kind $(C, C)_{1,2}$, $(C, S)_{1,2}$, $(C, K)_{1,2}$, $(S, K)_{1,2}$. It remains to analyze the cases $(Y, K)_{2,3}$ and (K, K).

Assume x is a $(Y, K)_2$ Frame Point $x = \gamma^+(\mathbf{T}(x))$ (being the case of γ^- entirely similar). For $\varepsilon > 0$ sufficiently small, there are two strips $\mathcal{S}^{1,2}$ containing $\gamma|[\mathbf{T}(x)-\varepsilon, \mathbf{T}(x)]$ and one strip $\mathcal{S}^3$ reaching x from the other side. We indicate by $\mathcal{R}_+$ the reachable set corresponding to trajectories of $\mathcal{S}^{1,2}$ and $\mathcal{R}_*$ that corresponding to trajectories of $\mathcal{S}^3$. Generically, we have that $Int(C^N(\mathcal{R}_+(\mathbf{T}(x)), x)) \neq \emptyset$ and $C^N(\mathcal{R}_*(\mathbf{T}(x)), x)$ is a half line. Hence x is the first point reached on the overlap curve K and is a minimum for $\mathbf{T}$ restricted to K. The front $F_{\mathbf{T}(x)}$ is formed by a smooth branch on the strip $\mathcal{S}^3$ and by a piecewise $\mathcal{C}^1$ branch with an angle at x on the strips $\mathcal{S}^{1,2}$. For times $t < \mathbf{T}(x)$, near $\mathbf{T}(x)$, the front F_t consists of one smooth branch on $\mathcal{S}^3$ and one piecewise smooth on $\mathcal{S}^{1,2}$, while for $t > \mathbf{T}(x)$, near $\mathbf{T}(x)$, consists of two piecewise smooth branches crossing K on opposite sides of x. We conclude that x is topologically a regular saddle for $\mathbf{T}$.

The analysis of the point $(Y, K)_3$ can be done in a similar way. Notice that x is an endpoint of K and one easily check that x is a minimum for $\mathbf{T}$ restricted to K. However, now there are four strips intersecting at x. Two of them ends at K coming from different directions, while the other two have the γ_k curve through x as border with γ^+ being the base of one of the two. Again, we conclude that x is topologically a regular saddle for $\mathbf{T}$.

Assume x is of kind (K, K). If x is reached by a γ_0 Frame Curve then the fronts on each side of K are $\mathcal{C}^1$ and generically we are in case a) above, where D now is the union of the two K curves, and $\mathbf{T}$ is topologically equivalent to a nonzero linear function. If x is reached by a γ_k Frame Curve, then the corresponding front has an angle on γ_k and the reachable set is locally concave, see Section 5.3.7. The function $\mathbf{T}$ has a maximum at x and x is topologically a regular maximum.

Thus for every $T > 0$, $\mathbf{T}|_{Int(\mathcal{R}(T))}$ is topologically a Morse function with the set Q being the collection of all Frame Curves in $Int(\mathcal{R}(T))$.

5.5 Semiconcavity for a two dimensional target on the plane

The aim of this section is to prove semiconcavity of the minimum time function for planar systems with a two dimensional manifold with boundary instead of a point as starting set.

Definition 34 *A function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^2$ open, is said to be semiconcave if for every compact convex $\mathcal{K} \subset A$, there exists $c_{\mathcal{K}} > 0$ such that $f(x) - c_{\mathcal{K}} |x|^2$ is concave on $\mathcal{K}$.*

If we fix a two dimensional manifold $\mathcal{T}$ with boundary as starting set then, from the transversality condition of Pontryagin Maximum Principle, the covectors of extremal trajectories at initial time must belong to the normal space to $\partial\mathcal{T}$. Moreover we assume the following local controllability property:

(LC) For every $x \in \partial\mathcal{T}$ there exists a control $u \in [-1, 1]$ such that $F(x) + uG(x)$ points inside $\mathcal{T}$.

Theorem 28 *Let $\mathcal{T}$ be a smooth two dimensional manifold with boundary and assume that* **(LC)** *holds. Then the minimum time function $\mathbf{T}_{\mathcal{T}}$, with initial set $\mathcal{T}$, is semiconcave.*

Proof. The assumption **(LC)** ensures that the minimum time function $\mathbf{T}_{\mathcal{T}}$ is continuous. The structure of extremal trajectories and singularities of extremal synthesis have been described in [18]. The main difference for the optimal synthesis is that now there are no more the $\gamma^{\pm}$ trajectories and the relative Frame Points. In an entirely similar manner as above, one can prove that generically the minimum time function is topologically a Morse function and the Minimum Time Front has the same properties as before.

Obviously a smooth function is semiconcave, hence we have only to consider Frame Curves and Frame Points. It is easy to check that points, at which $\mathbf{T}_{\mathcal{T}}$ is $\mathcal{C}^1$ and topologically equivalent to a linear function, satisfy the assumptions of semiconcavity. The same happens for maxima.

Thus we have only to check minima and saddles. The only minimum of the starting point case is the starting point itself and now it is replaced by $\partial\mathcal{T}$; there is no other minimum. Moreover, from the previous analysis we know that the possible saddle points are internal points of overlap curves K or Frame Points of kind $(Y, K)_{2,3}$. In the first case the function $\mathbf{T}_{\mathcal{T}}$ is smooth along K and again we retrieve semiconcavity, while the second case does not happen (there are no $\gamma^{\pm}$ curves now). ∎

Chapter 6

Application to Second Order Differential Equation

6.1 Statment of the Problem

The program developed in [17, 19, 31, 32] and in this Thesis relatively to the minimum time stabilization to the origin of the control system:

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad F, G \ \in \ \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2), \quad F(0) = 0, \quad |u| \leq 1, \tag{6.1}$$

can be used to classify the solutions to the following problem:

Problem: *Consider the autonomous ODE in $\mathbb{R}$:*

$$\ddot{y} \ = \ f(y, \dot{y}), \tag{6.2}$$

$$f \in \mathcal{C}^3(\mathbb{R}^2), \qquad f(0,0) = 0 \tag{6.3}$$

that describes the motion of a point under the action of a force depending on the position and velocity of the point (for instance due to a magnetic field or the viscosity of a fluid). Let us apply an external force, that we suppose bounded (e.g. $|u| \leq 1$):

$$\ddot{y} = f(y, \dot{y}) + u. \tag{6.4}$$

Controlling the external force we want to reach in minimum time a point (y_0, v_0), of the configuration space, from the rest state $(0,0)$.

First of all observe that if we set $x_1 = y$, $x_2 = \dot{y}$, (6.4) becomes:

$$\dot{x}_1 \ = \ x_2 \tag{6.5}$$

$$\dot{x}_2 \ = \ f(x_1, x_2) + u, \tag{6.6}$$

that can be written in our standard form $\dot{x} = F(x) + uG(x)$, $x \in \mathbb{R}^2$ by setting $x = (x_1, x_2)$, $F(x) = (x_2, f(x))$, $G(x) = (0, 1) := G$.
In this case we have:

$$\Delta_A(x) = \det(F(x), G(x)) = x_2 \tag{6.7}$$
$$\Delta_B(x) = \det(G(x), [F(x), G(x)]) = 1. \tag{6.8}$$

From these it follows:

$$\Delta_A^{-1}(0) = \{x \in \mathbb{R}^2 : x_2 = 0\} \tag{6.9}$$
$$\Delta_B^{-1}(0) = \emptyset. \tag{6.10}$$

By direct computations it is easy to see that the *generic* conditions **P1**,...,**P9** (see Chapter 1), under which the construction of [31] holds, are satisfied under the condition:

$$f(x_1, 0) = \pm 1 \quad \Rightarrow \quad \partial_1 f(x_1, 0) \neq 0 \tag{6.11}$$

that obviously implies $f(x_1, 0) = 1$ or $f(x_1, 0) = -1$ only in a finite number of points.
In the framework of [17, 31, 32, 19] we will prove that, for our problem (6.5), (6.6), with the condition (6.11), the "shape" of the optimal synthesis is that shown in figure 6.1.

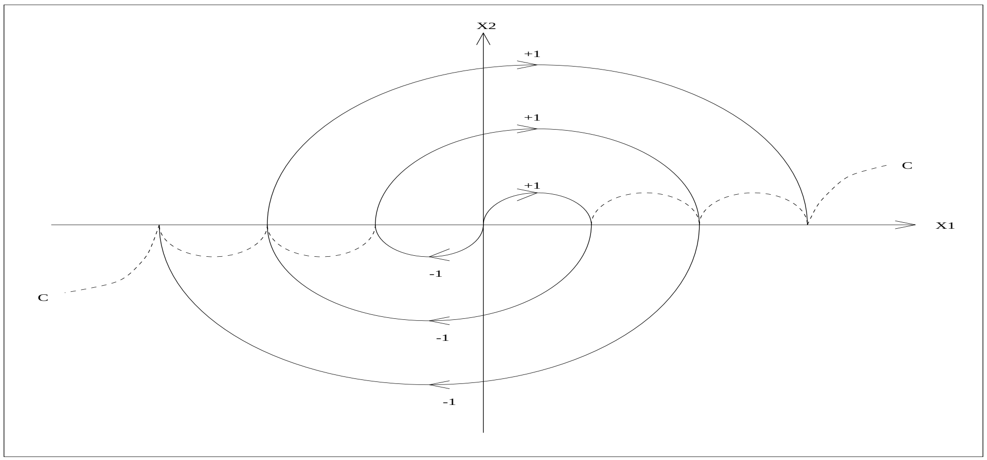


Figure 6.1: The shape of the optimal synthesis for our problem

In particular the partition of the reachable set is described by the following:

Theorem 29 *The optimal synthesis of the control problem (6.5) (6.6) with the condition (6.11), satisfies the following:*

1. *there are no turnpikes;*

2. *the trajectory $\gamma^{\pm}$ (starting from the origin and corresponding to constant control ± 1) exits the origin with tangent vector $(0, \pm 1)$ and, for an interval of time of positive measure, lies in the set $\{(x_1, x_2) : \ x_1, x_2 \geq 0\}$ respectively $\{(x_1, x_2) : \ x_1, x_2 \leq 0\}$;*

3. *$\gamma^{\pm}$ is optimal up to the first intersection (if it exists) with the x_1-axis. At the point in which γ^{+} intersects the x_1-axis it generates a switching curve that lies in the half plane $\{(x_1,x_2): \ x_2 \geq 0\}$ and ends at the next intersection with the x_1-axis (if it exists). At that point another switching curve generates. The same happens for γ^{-} and the half plane $\{(x_1,x_2): \ x_2 \leq 0\}$;*

4. *let y_i, for $i = 1,...,n$ (n possibly $+\infty$) (respectively z_i, for $i = 1,...m$ (m possibly $+\infty$)) be the set of boundary points of the switching curves contained in the half plane $\{(x_1,x_2): \ x_2 \geq 0\}$ (respectively $\{(x_1,x_2): \ x_2 \leq 0\}$) ordered by increasing (resp. decreasing) first components. Under generic assumptions, y_i and z_i do not accumulate. Moreover:*

- *For $i = 2,...,n$, the trajectory corresponding to constant control +1 ending at y_i starts at z_{i-1};*
- *For $i = 2,...,m$, the trajectory corresponding to constant control -1 ending at z_i starts at y_{i-1}.*

<u>Observation</u> **6** The union of $\gamma^{\pm}$ with the switching curves is a one dimensional $\mathcal{C}^0$ manifold M. Above this manifold the optimal control is $+1$ and below is -1.

<u>Observation</u> **7** The optimal trajectories turn clockwise around the origin and switch along the switching part of M . They stop turning after the last y_i or z_i and tend to infinity with $x_1(t)$ monotone after the last switching.

From **4** of Theorem 29 it follows immediately the following:

Theorem 30 *To every optimal synthesis for a control problem of the type (6.5) (6.6) with the condition (6.11), it is possible to associate a couple $(n,m) \in (\mathbb{N} \cup \infty)^2$ such that one of the following cases occurs:*

A. *$n = m$, n finite;*

B. *$n = m + 1$, n finite;*

C. *$n = m - 1$, n finite;*

D. *$n = \infty, \quad m = \infty$.*

Moreover, if Γ_1, Γ_2 are two optimal syntheses for two problems of kind (6.5), (6.6), (6.11), and (n_1,m_1) (resp. (n_2,m_2)) are the corresponding couples, then Γ_1 is equivalent to Γ_2 iff $n_1 = n_2$ and $m_1 = m_2$.

Observation **8** In Theorem 30 the equivalence between optimal syntheses is the one defined in [17] (see also Definition 19).

Proof of Theorem 29 Let us consider the synthesis constructed by the algorithm described in [17]. The stability assumptions **(GA1)**–**(GA9)** holds. The optimality follows from Theorem 3.1 of [17].

1. By definition a turnpike is a subset of $\Delta_B^{-1}(0)$. From (6.8) it follows the conclusion.

2. We leave the proof to the reader.

3. Let $\gamma_2^{\pm}(t) = \pi_2(\gamma^{\pm}(t))$, where $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_2(x_1, x_2) = x_2$, and consider the adjoint vector field $v : \mathbb{R}^2 \times \mathrm{Dom}(\gamma^{\pm}) \times \mathrm{Dom}(\gamma^{\pm}) \to \mathbb{R}^2$ associated to $\gamma^{\pm}$ that is the solution of the Cauchy problem:

$$\begin{cases} \dot{v}(v_0, t_0; t) & = & (\nabla F \pm \nabla G)(\gamma^{\pm}(t)) \cdot v(v_0, t_0; t) \\ v(v_0, t_0; t_0) & = & v_0, \end{cases} \tag{6.12}$$

We have the following:

Lemma 6 *Consider the $\gamma^{\pm}$ trajectories for the control problem (6.5), (6.6). We have that $v(G, t; 0)$ and G are parallel iff $\Delta_A(\gamma^{\pm}(t)) = 0$ (i.e. $\gamma_2^{\pm}(t) = 0$).*

Proof Consider the curve γ^+, the case of γ^- being similar. From (6.9) we know that $\Delta_A(\gamma^+(t)) = 0$ iff $\gamma_2^+(t) = 0$. First assume $\Delta_A(\gamma^+(t)) = 0$. We have that G and $(F + G)(\gamma^+(t))$ are collinear that is $G = \alpha(F + G)(\gamma^+(t))$ with $\alpha \in \mathbb{R}$. For fixed t_0, t the map:

$$f_{t_0,t} : v_0 \mapsto v(v_0, t_0; t) \tag{6.13}$$

is clearly linear and injective, then using (6.12) and $\dot{\gamma}^+(t) = (F + G)(\gamma^+(t))$, we obtain $v(G, t; 0) = \alpha v\left((F + G)(\gamma^+(t)), t; 0\right) = \alpha(F + G)(0) = \alpha G$.
Viceversa assume $v(G, t; 0) = \alpha G$, then (as above) we obtain $v(G, t; 0) = \alpha v((F + G)(\gamma^+(t)), t; 0)$. From the linearity and the injectivity of (6.13) we have $G = \alpha(F + G)(\gamma^+(t))$ hence $\Delta_A(\gamma^+(t)) = 0$. ∎

Lemma 7 *Consider the trajectory γ^+ for the control problem (6.5), (6.6). Let $\bar{t} > 0$ (possibly $+\infty$) be the first time such that $\gamma_2^+(\bar{t}) = 0$. Then γ^+ is extremal exactly up to time $\bar{t}$. And similarly for γ^-.*

Proof Consider the function $\theta(t) = \arg\left(G(0), v\left(G(\gamma^+(t)), t, 0\right)\right)$, defined in Chapter 1 (cfr. formula (1.10)) This function has the following properties:

(i) $\mathrm{sgn}(\dot{\theta}(t)) = \mathrm{sgn}(\Delta_B(\gamma(t))$, that was proved in Lemma 3.4 of [31]. From (6.8) we have that $\mathrm{sgn}(\dot{\theta}(t)) = 1$ so $\theta(t)$ is strictly increasing;

(ii) γ^+ is extremal exactly up to the time in which the measure of the range of θ is π i.e. up to the time:

$$t^+ = \min\{t \in [0, \infty] : |\theta(s_1) - \theta(s_2)| = \pi, \text{ for some } s_1, s_2 \in [0, t^+]\}, \quad (6.14)$$

under the hypothesis $\dot{\theta}(t^+) \neq 0$. This was proved in proposition 3.1 of [17].

From Lemma 6 we have that $\Delta_A(\gamma^+(t)) = 0$ iff there exists $n \in \mathbf{N}$ satisfying:

$$\theta(G, v(G, t, 0)) = n\pi. \quad (6.15)$$

In particular (6.15) holds for $t = \bar{t}$ and some n. From the fact that $\bar{t}$ is the first time in which $\gamma_2^+(\bar{t}) = 0$ and hence the first time in which $\Delta_A(\gamma^+(\bar{t})) = 0$, we have that $n = 1$.

From $\theta(\bar{t}) = \pi$ and $\mathrm{sgn}(\dot{\theta}(\bar{t})) = 1$ the Theorem is proved with $t^+ = \bar{t}$. ■

From Lemma 7, $\gamma^\pm$ are extremal up to the first intersection with the x_1-axis.

Let $\underline{t}$ be the time such that $\gamma^-(\underline{t}) = z_1$, defined in **4** of Theorem 29. The extremal trajectories that switch along the C-curve starting at y_1 (if it exists), are the trajectories that start from the origin with control -1 and then, at some time $t' < \underline{t}$, switch to control $+1$. Since the first switching occurs in the orthant $\{(x_1, x_2) : \quad x_1, x_2 < 0\}$, by a similar argument to the one of Lemma 7, the second switch has to occur in the half space $\{(x_1, x_2) : \quad x_2 > 0\}$, because otherwise between the two switches we have $\mathrm{meas}(\mathrm{range}(\theta(t))) > \pi$. This proves that the switching curves never cross the x_1-axis.

4. The two assertions can be proved separately. Let us demonstrate only the first, being the proof of the second similar. Define $y_0 = z_0 = (0, 0)$. By definition the $+1$ trajectory starting at z_0 reaches y_1. By Theorem 13 (Chapter 3), we know that if an extremal trajectory has a switching at a point of the x_1–axis, then it switches iff it intersects the x_1–axis again. This means that the extremal trajectory that switches at y_i has a switching at z_j for some j. By induction one has $j \geq i - 1$. Let us prove that $j = i - 1$. By contradiction assume that $j > i - 1$, then there exists an extremal trajectory switching at z_{i-1} that switches on the C curve with boundary points y_{i-1}, y_i. This is forbidden by Theorem 13 (Chapter 2). ■

6.2 Examples

In the following we will show the qualitative shape of the synthesis of some physical systems coupled with a control. More precisely we want to determine the value of the couple (m, n) of Theorem 30.

6.2.1 Duffin Equation

The Duffin equation is given by the formula $\ddot{y} = -y - \varepsilon(y^3 + 2\mu\dot{y}), \quad \varepsilon, \mu > 0, \quad \varepsilon$ small. By introducing a control term and transforming the second order equation in a first order system, we have:

$$\dot{x}_1 = x_2 \tag{6.16}$$

$$\dot{x}_2 = -x_1 - \varepsilon(x_1^3 + 2\mu x_2) + u. \tag{6.17}$$

From this form it is clear that $f(x) = -x_1 - \varepsilon(x_1^3 + 2\mu x_2)$.
Consider the trajectory γ^+. It starts with tangent vector $(0, 1)$, then, from (6.16), we see that it moves in the orthant $\Omega := \{(x_1, x_2), \quad x_1, x_2 > 0\}$. To know the shape of the synthesis we need to know where $(F+G)_2(x) = 0$. If we set $a = \frac{1}{2\varepsilon\mu}$, this happens where

$$x_2 = a(1 - x_1 - \varepsilon x^3). \tag{6.18}$$

From (6.16) and (6.17) we see that, after meeting this curve, the trajectory moves with $\dot{\gamma}_1^+ > 0$ and $\dot{\gamma}_2^+ < 0$. Then it meets the x_1-axis because otherwise if $\gamma^+(t) \in \Omega$ we necessarily have (for $t \to \infty$) $\gamma_1^+ \to \infty$, $\dot{\gamma}_2^+ \to 0$, that is not permitted by (6.17). The behavior of the trajectory γ^- is similar.

In this case, the numbers (n, m) are clearly (∞, ∞) because the $+1$ trajectory that starts at z_1 meets the curve (6.18) exactly one time and behaves like γ^+. So the C-curve that starts at y_1 meets again the x_1 axis. The same happen for the -1 curve that starts at y_1. In this way an infinite sequence of y_i and z_i is generated.

6.2.2 Van der Pol equation

The Van der Pol equation is given by the formula $\ddot{y} = -y + \epsilon(1 - y^2)\dot{y} + u, \quad \varepsilon > 0$ and small. The associated control system is: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$. We have $(F+G)_2(x) = 0$ on the curves $x_2 = -\frac{1}{\varepsilon(x_1+1)}$ for $x_1 \neq \pm 1$, $x_1 = -1$. After meeting these curves, the γ^+ trajectory moves with $\dot{\gamma}_1^+ > 0$ and $\dot{\gamma}_2^+ < 0$ and, for the same reason as before, meets the x_1-axis. Similarly for γ^-. As in the Duffin equation, we have that m and n are equal to $+\infty$. But here, starting from the origin, we cannot reach the regions: $\{(x_1, x_2) : \ x_1 < -1, x_2 \geq -\frac{1}{\varepsilon(x_1+1)}\}$, $\{(x_1, x_2) : \ x_1 > -1, x_2 \leq -\frac{1}{\varepsilon(x_1-1)}\}$ (see figure 6.2).

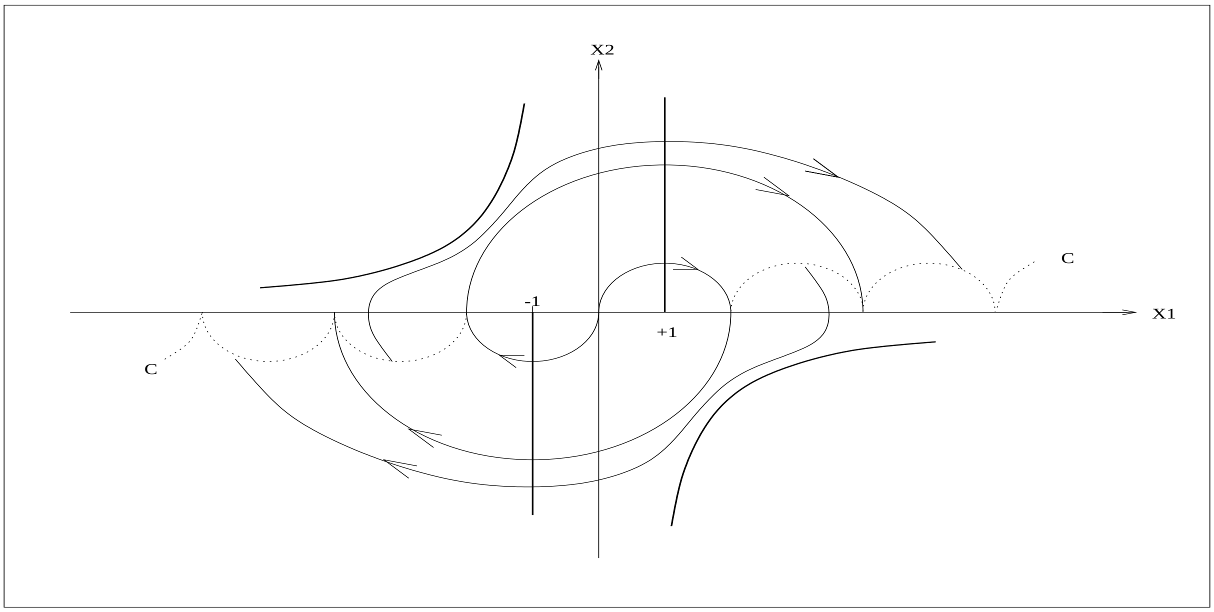


Figure 6.2: The synthesis for the Van Der Pol equation.

6.2.3 Another example

In the following we will study an equation whose synthesis has $n, m < \infty$. Consider the equation: $\ddot{y} = -e^y + \dot{y} + 1$. The associated control system is:

$$\dot{x}_1 = x_2 \tag{6.19}$$
$$\dot{x}_2 = -e^{x_1} + x_2 + 1 + u \tag{6.20}$$

We have $\dot{\gamma}_2^+ = 0$ on the curve $x_2 = e^{x_1} - 2$. After meeting this curve, the γ^+ trajectory meets the x_1-axis.

Now the synthesis has a different shape because the trajectories corresponding to control -1 satisfies $\dot{\gamma}_2 = 0$ on the curve:

$$x_2 = e^{x_1} \tag{6.21}$$

that is contained the half plane $\{(x_1, x_2) : \quad x_2 > 0\}$. Hence γ^- never meets the curve given by (6.21) and this means that $m = 0$. Since we know that n is at least 1, by observation 4, we have $n = 1, \quad m = 0$. The synthesis is drawn in figure 6.3.

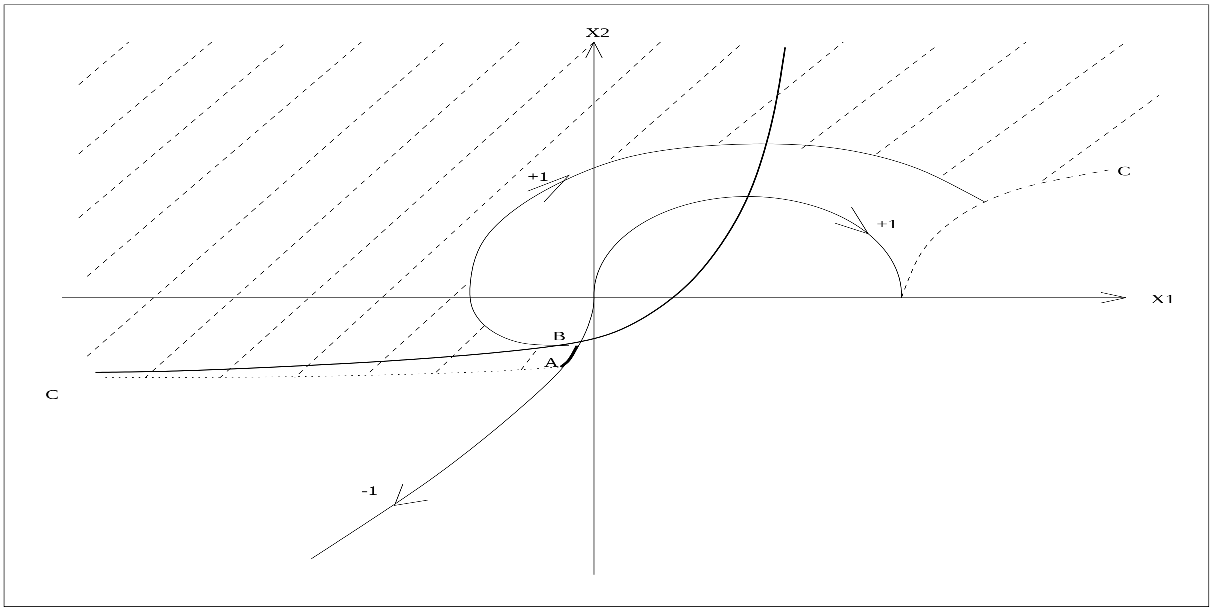


Figure 6.3: The synthesis for the control problem (6.19), (6.20). The sketched region is reached by curves that start from the origin with control -1 and then switch to $+1$ control between the points A and B.

6.3 Optimal syntheses for Bolza Problems

Quite easily we can adapt the previous program to obtain information about the optimal syntheses associated (in the previous sense) to second order differential equations, but for more general minimizing problems.

We have the well known:

Lemma 8 *Consider the control system:*

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad F, G \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R}^2), \quad F(0) = 0, \quad |u| \leq 1. \tag{6.22}$$

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^3$ bounded function such that there exists $\delta > 0$ satisfying $L(x) > \delta$ for any $x \in \mathbb{R}^2$.
Then, for every $x_0 \in R^2$, the problem: $\min \int_0^\tau L(x(t))dt$ *s.t.* $x(0) = 0, \quad x(\tau) = x_0$, *is equivalent to the* minimum time problem *(with the same boundary conditions) for the control system $\dot{x} = F(x)/L(x) + uG(x)/L(x)$.*

By this Lemma it is clear that if we have a second order differential equation with a bounded–external force $\ddot{y} = f(y, \dot{y}) + u$, $f \in \mathcal{C}^3(\mathbb{R}^2)$, $f(0,0) = 0$, $|u| \leq 1$, then the problem of reaching a point in the configuration space (y_0, v_0) from the origin, minimizing

$\int_0^T L(y(t), \dot{y}(t))dt$, (under the hypotheses of Lemma 8) is equivalent to the minimum time problem for the system: $\dot{x}_1 = x_2/L(x)$, $\dot{x}_2 = f(x)/L(x) + 1/L(x)u$. By setting: $\alpha : \mathbb{R}^2 \to]0, 1/\delta[$, $\alpha(x) := 1/L(x)$, $\beta : \mathbb{R}^2 \to \mathbb{R}$, $\beta(x) := f(x)/L(x)$, we have: $F(x) = (x_2\alpha(x), \beta(x))$, $G(x) = (0, \alpha(x))$. From these it follows: $\Delta_A(x) = x_2\alpha^2$, $\Delta_B(x) = \alpha^2(\alpha + x_2\partial_2\alpha)$.
The equations defining turnpikes are: $\Delta_A \neq 0$, $\Delta_B = 0$, that with our expressions become the differential condition $\alpha + x_2\partial_2\alpha = 0$ that in terms of L is:

$$L(x) - x_2\partial_2 L(x) = 0 \tag{6.23}$$

<u>*Observation*</u> **9** Since $L > 0$ it follow that the turnpikes never intersect the x_1-axis. Since (6.23) depends on $L(x)$ and not on the control system, all the properties of the turnpikes depend only on the Lagrangian.

Now we consider some particular cases of Lagrangians.

L=$L(y)$ In this case the Lagrangian depends only on the position y and not on the velocity $\dot{y}$ (i.e. $L = L(x_1)$). (6.23) is never satisfied so there are no turnpikes.

L=$L(\dot{y})$ In this case the Lagrangian depends only on velocity and the turnpikes are horizontal lines.

L=$V(y) + \frac{1}{2}\dot{y}^2$ In this case we want to minimize an energy with a kinetic part $\frac{1}{2}\dot{y}^2$ and a positive potential depending only on the position and satisfying $V(y) > 0$. The equation for turnpikes is $(x_2)^2 = 2V(x_1)$.

Chapter 7

Stability of Planar Switched Systems

7.1 Statment of the Main Results

Given two diagonalizable 2×2 real matrices with eigenvalues having strictly negative real part, we consider the following property:

($\mathcal{P}$) The dynamical system in $\mathbb{R}^2$:

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{7.1}$$

is asymptotically stable at the origin for each measurable function $u : [0, \infty[\to [0, 1]$.

In this Section we state the necessary and sufficient conditions on A and B under which ($\mathcal{P}$) holds, under the the hypothesis that they are diagonalizable and at least one of the two matrices has complex eigenvalues. The case in which both A and B have real eigenvalues is similar and it is treated in [10]. Moreover we study under which conditions we have at least stability for each function $u(.)$.

Set $M(u) := uA + (1 - u)B$. In the class of constant functions the asymptotic stability of the origin of the system (7.1) occurs iff the matrix $M(u)$ has eigenvalues with strictly negative real part for each $u \in [0, 1]$ so that for the class of measurable functions this is only a necessary condition. On the other hand it is know that if $[A, B] = 0$ then the system (7.1) is asymptotically stable for each function $u(.)$. Let λ_1, λ_2 (resp. λ_3, λ_4) be the eigenvalues of A (resp. B). The condition $[A, B] \neq 0$ implies $\lambda_1 \neq \lambda_2$ and $\lambda_3 \neq \lambda_4$. In the following we assume always the conditions $Re(\lambda_1),\ Re(\lambda_2),\ Re(\lambda_3),\ Re(\lambda_4) < 0,\ \lambda_1 \neq \lambda_2,\ \lambda_3 \neq \lambda_4$. Moreover we say that $\gamma : [0, \infty[\to \mathbb{R}^2$ is a trajectory of the switched system (7.1) if there exists $u : [0, \infty[\to [0, 1]$ such that for almost every $t \in [0, \infty[$ it holds $\dot{\gamma}(t) = u(t)A\gamma(t) + (1 - u(t))B\gamma(t)$.

Theorems 31 below (proved in the next section), gives necessary and sufficient conditions for the stability of the system (7.1) in terms of three (coordinates invariant) parameters defined in Definition 35. The first (ρ_A) depends on the eigenvalues of A, the second (ρ_B) depends on the eigenvalues of B and the third ($\mathcal{K}$) depends on $Tr(AB)$ which is a standard scalar product in the space of 2×2 matrices. Proposition 20 gives some properties of this parameters. Finally Proposition 21 shows the geometric meaning of $\mathcal{K}$. In the case $[A,B]\neq 0$ (that is the case in which we are interested in) it is in 1–1 correspondence with the cross ratio of the four points in the projective line $\mathbb{C}P^1$ that correspond to the four eigenvectors of A and B. This parameter contains the interrelation between the two systems.

Definition 35 *Let A and B be two 2×2 real matrices. Suppose that they are diagonalizable and that at least A or B has complex eigenvalues. Let λ_1,λ_2 (resp. λ_3,λ_4) be the eigenvalues of A (resp. B), and suppose $Re(\lambda_1),\ Re(\lambda_2),\ Re(\lambda_3),\ Re(\lambda_4)<0,\ \lambda_1\neq\lambda_2,\ \lambda_3\neq\lambda_4$. Moreover choose the labels $(_1)$ and $(_2)$ (resp. $(_3)$ and $(_4)$) in such a way $|\lambda_2|>|\lambda_1|$ (resp. $|\lambda_3|>|\lambda_4|$) if they are real or $Im(\lambda_2)<0$ (resp. $Im(\lambda_4)<0$) if they are complex. Define:*

$$\rho_A:=-i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2};\qquad \rho_B:=-i\frac{\lambda_3+\lambda_4}{\lambda_3-\lambda_4};\qquad \mathcal{K}:=2\frac{Tr(AB)-\frac{1}{2}Tr(A)Tr(B)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}.\tag{7.2}$$

Moreover define the following function of $\rho_A,\rho_B,\mathcal{K}$:

$$\mathcal{D}:=\mathcal{K}^2+2\rho_A\rho_B\mathcal{K}-(1+\rho_A^2+\rho_B^2);\tag{7.3}$$

Notice that $\rho_A\in\mathbb{R}$, $\rho_A>0$ iff A has complex eigenvalues and $\rho_A\in i\mathbb{R}$, $\rho_A/i>1$ iff A has real eigenvalues. The same holds for B. Moreover $\mathcal{D}\in\mathbb{R}$. The parameter $\mathcal{K}$ contains important information about the matrices A and B. They are state in the following:

Proposition 20 *Let A and B be as in Definition 35. We have the following:*

- *$[A,B]=0$ iff $\mathcal{K}\in\mathbb{R}$ and $|\mathcal{K}|=1$. In this case A and B have both complex eigenvalues;*
- *A and B have both complex eigenvalues with $[A,B]\neq 0$ iff $\mathcal{K}\in\mathbb{R}$ and $|\mathcal{K}|>1$;*
- *A and B have one complex and the other real eigenvalues iff $\mathcal{K}\in i\mathbb{R}$;*

Proof For later use we prefer to treat the case in which both matrices have complex eigenvalues (in the following **(CC)**) and the case in which one of the two matrices has real and the other complex eigenvalues (in the following **(RC)**) separately. In the **(CC)** case, choose a system of coordinates in which:

$$A = \begin{pmatrix} -\delta_A & -\omega_A/E \\ \omega_A E & -\delta_A \end{pmatrix} \qquad B = \begin{pmatrix} -\delta_B & -\omega_B \\ \omega_B & -\delta_B \end{pmatrix},$$

where $E \in \mathbb{R} \setminus \{0\}$, $\delta_A, \delta_B, \omega_A, \omega_B > 0$. We have $\mathcal{K} = \frac{1}{2}(E + 1/E)$ and:

$$[A, B] = \omega_A \omega_B (E - 1/E) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

from which it follows $\mathcal{K} \in \mathbb{R}, \quad |\mathcal{K}| \geq 1$ and $[A, B] = 0$ iff $\mathcal{K} = \pm 1$.

In the **(RC)** case, define:

$$R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \in SO(2), \tag{7.4}$$

and choose a system of coordinates in which:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad B = R^{-1}(\varphi) \begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix} R(\varphi),$$

where $\varphi \in [0, \pi/2[, \quad E \in \mathbb{R} \setminus \{0\}$. In this case we have $\mathcal{K} = i(E - 1/E)\cos(\varphi)\sin(\varphi) \in i\mathbb{R}$ and

$$[A, B] = (\lambda_1 - \lambda_2) \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \neq 0 \text{ for each } \mathcal{K} \in i\mathbb{R}.$$

More details are given in Subsection 7.2.2. This concludes the proof. ∎

Since we are interested in the case $[A, B] \neq 0$, in the following we assume $\mathcal{K} \in (\mathbb{R} \setminus \{-1, 1\}) \cup i\mathbb{R}$.

Theorem 31 *Let A and B be two diagonalizable non commuting 2×2 real matrices with eigenvalues having real part strictly less than 0. Suppose that at least one of the two matrices has complex eigenvalues and define $\rho_A, \rho_B, \mathcal{K}, \mathcal{D}$ as in Definition 35. We have the following stability conditions:*

Case (CC) *If $\mathcal{K} \in \mathbb{R}, \quad |\mathcal{K}| > 1$ (that is the case in which A and B have both complex eigenvalues) then:*

- **Case (CC.1)** *if $\mathcal{D} < 0$ then $(\mathcal{P})$ is true;*
- **Case (CC.2)** *if $\mathcal{D} > 0$ then:*
  - **Case (CC.2.1)** *if $\mathcal{K} < -1$ then $(\mathcal{P})$ is false;*

Case (CC.2.2) *if $\mathcal{K} > 1$ then then $(\mathcal{P})$ is true iff it holds the following condition:*

$$\begin{aligned}\rho_{CC} \ &:= \ \exp\left(-\rho_A \arctan\left(\frac{-\rho_A\mathcal{K}+\rho_B}{\sqrt{\mathcal{D}}}\right) - \rho_B \arctan\left(\frac{\rho_A-\rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right) - \right. \\ &\qquad \left.\frac{\pi}{2}(\rho_A+\rho_B)\right) \times \\ &\sqrt{\frac{(\rho_A\rho_B+\mathcal{K})+\sqrt{\mathcal{D}}}{(\rho_A\rho_B+\mathcal{K})-\sqrt{\mathcal{D}}}} < 1 \end{aligned} \tag{7.5}$$

Case (CC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ is true or false according respectively to the fact that $\mathcal{K} > 1$ or $\mathcal{K} < -1$.*

Case (RC) *If $\mathcal{K} \in i\mathbb{R}$ (that is the case in which A and B have one complex and the other real eigenvalues), define $\chi := \rho_A\mathcal{K} - \rho_B$ where ρ_A and ρ_B are chosen in such a way $\rho_A \in i\mathbb{R}$, $\rho_B \in \mathbb{R}$. Then:*

Case (RC.1) *if $\mathcal{D} > 0$ then $(\mathcal{P})$ is true;*

Case (RC.2) *if $\mathcal{D} < 0$ then $\chi \neq 0$ and we have:*

Case (RC.2.1) *if $\chi > 0$ then (P) is false. Moreover in this case $\mathcal{K}/i < 0$;*

Case (RC.2.2) *if $\chi < 0$, then:*

Case (RC2.2.A) *if $\mathcal{K}/i \leq 0$ then $(\mathcal{P})$ is true;*

Case (RC2.2.B) *if $\mathcal{K}/i > 0$ then $(\mathcal{P})$ is true iff it holds the following condition:*

$$\rho_{RC} := e^{-\rho_B(\xi^+-\xi^-)}\sqrt{\frac{\cos^2\xi^+ + E^2\sin^2\xi^+}{\cos^2\xi^- + E^2\sin^2\xi^-}} \times$$
$$\sqrt{\cos^2\theta^+\left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i+1)} + \sin^2\theta^+\left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i-1)}} < 1 \tag{7.6}$$

where:

$$E := \mathcal{K}/i + \sqrt{-\mathcal{K}^2+1}$$
$$m^{\pm} := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i-1)\mathcal{K}/i}$$
$$\theta^+ := \arctan m^+ \tag{7.7}$$
$$\xi^{\pm} := \arctan\left(\frac{m^{\pm}-1}{E(m^{\pm}+1)}\right), \quad \xi^+ \in]\xi^-, \xi^- + \pi[. \tag{7.8}$$

Case (RC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ is true or false according respectively to the fact that $\chi < 0$ or $\chi > 0$.*

Finally if $(\mathcal{P})$ is false then in case **CC.2.2** *with* $\rho_{CC} = 1$, *case* **RC.2.2.B** *with* $\rho_{RC} = 1$, *case* **(CC.3)** *with* $\mathcal{K} < -1$ *and case* **(RC.3)** *with* $\chi > 0$, *for every* $C > 0$, *there exists* $C' \le C$ *such that if* $|\gamma(0)| < C'$ *then* $|\gamma(t)| < C$ *for every* $t \in [0, \infty[$ *(i.e. we have stability of the origin). In the other cases there exists a trajectory* $\gamma(t)$ *such that* $\lim_{t\to\infty} |\gamma(t)| = \infty$.

Notice that the expression (7.5) is invariant if we exchange ρ_A with ρ_B. The last statement say when we have at least stability for every trajectory γ of the system (7.1).

Let us study the geometric meaning of $\mathcal{K}$. Let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$ belong to the complex projective line $\mathbb{C}P^1$. Suppose $\mathbf{V}_1 \neq \mathbf{V}_2 \neq \mathbf{V}_3$ and let (v_1, v_1'), (v_2, v_2'), (v_3, v_3'), (v_4, v_4') be the corresponding homogeneous coordinates. The *cross ratio* $\beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4)$ is defined in the following way. Make a Moebious transformation such that $\mathbf{V}_1, \mathbf{V}_2$ became the fundamental points (i.e. of homogeneous coordinates respectively $(0, 1)$ and $(1, 0)$) and $\mathbf{V}_3$ the unity point (i.e. of homogeneous coordinates $(1, 1)$) and let $(\bar{v}_4, \bar{v}_4')$ be the new homogeneous coordinates of $\mathbf{V}_4$. By definition we have:

$$\beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4) := \bar{v}_4'/\bar{v}_4 = \frac{\begin{vmatrix} v_1 & v_4 \\ v_1' & v_4' \end{vmatrix} \begin{vmatrix} v_2 & v_3 \\ v_2' & v_3' \end{vmatrix}}{\begin{vmatrix} v_2 & v_4 \\ v_2' & v_4' \end{vmatrix} \begin{vmatrix} v_1 & v_3 \\ v_1' & v_3' \end{vmatrix}} \tag{7.9}$$

Now the four eigenvectors of A and B are exactly four direction in $\mathbb{C}^2$ i.e. they can be regarded as four points of $\mathbb{C}P^1$. Moreover under the hypothesis $\lambda_1 \neq \lambda_2$ and $\lambda_3 \neq \lambda_4$ we have $\mathbf{V}_1 \neq \mathbf{V}_2$ and $\mathbf{V}_3 \neq \mathbf{V}_4$ and if we suppose $[A, B] \neq 0$ we have $\mathbf{V}_1 \notin \{\mathbf{V}_3, \mathbf{V}_4\}$ and $\mathbf{V}_2 \notin \{\mathbf{V}_3, \mathbf{V}_4\}$. So it make sense to compute their cross ratio.

We have the following:

Proposition 21 *Let A and B be two diagonalizable non commuting 2×2 real matrices with eigenvalues having strictly negative real part. Suppose that at least one of the two matrices has complex eigenvalues and let $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$ be the four point in the space $\mathbb{C}P^1$ corresponding respectively to the four eigenvectors of A and B chosen in such a way they correspond respectively to $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (see Definition 35). Let $\beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4)$ be their cross ratio and $\mathcal{K}$ the quantity defined in Definition 35. Then $\beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4)$ and $\mathcal{K}$ are in the one to one relation from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$:*

$$\mathcal{K} = \frac{\beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4) + 1}{\beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4) - 1}, \qquad \beta(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4) = \frac{\mathcal{K} + 1}{\mathcal{K} - 1}.$$

Proof We treat separately the two cases in which A and B have both complex and one real and the other complex eigenvalues using the same systems of coordinates of the proof of Proposition 20. The only difference is that now we have to assume $E \neq \pm 1$ in the **(CC)** cases (otherwise $[A,B]=0]$).

- In the **(CC)** case we have:

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ -iE \end{pmatrix} \quad \mathbf{V}_2 = \begin{pmatrix} 1 \\ +iE \end{pmatrix} \quad \mathbf{V}_3 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \mathbf{V}_4 = \begin{pmatrix} 1 \\ +i \end{pmatrix},$$

  and $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4) = (1+E)^2/(-1+E)^2$. Being $\mathcal{K} = \frac{1}{2}(E+1/E)$ we are done.

- In the **(RC)** case we have:

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{V}_3 = R^{-1}(\varphi)\begin{pmatrix} 1 \\ -iE \end{pmatrix} \quad \mathbf{V}_4 = R^{-1}(\varphi)\begin{pmatrix} 1 \\ +iE \end{pmatrix},$$

  and

$$\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4) = \frac{(i\,E\,\cos(\varphi)+\sin(\varphi))\;(-\cos(\varphi)-i\,E\,\sin(\varphi))}{(-i\,E\,\cos(\varphi)+\sin(\varphi))\;(-\cos(\varphi)+i\,E\,\sin(\varphi))}.$$

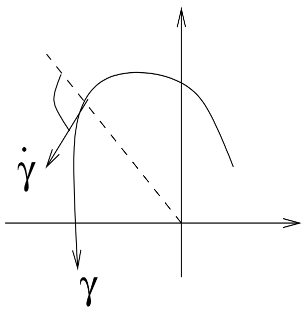
  From $\mathcal{K} = i(E-1/E)\cos(\varphi)\sin(\varphi)$ we have the conclusion. Proposition 21 is proved. ■

Notice that $\mathcal{K} \neq \infty$ so that $\beta \neq 1$. From Proposition 21 and Definition 35 we have the following expression for the extended cross ratio of the eigenvectors of A and B:

$$\beta = \frac{Tr({}^tAB) - (\lambda_1\lambda_4 + \lambda_2\lambda_3)}{Tr({}^tAB) - (\lambda_1\lambda_3 + \lambda_2\lambda_4)}$$

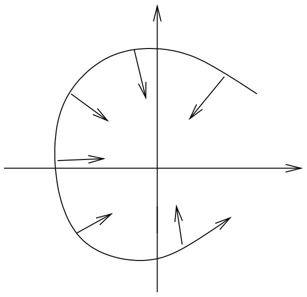
Theorem 31 is proved in the next session. Here we describe the main idea of the proof.

In both cases (**CC**) and (**RC**), we build the "worst trajectory" i.e. the trajectory that at each point has the velocity forming the angle, with the (exiting) radial direction, having the smallest absolute value, without taking care of the module of the velocity.

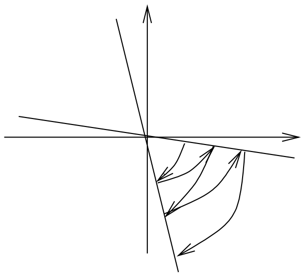


The main idea is that the system (7.1) is asymptotically stable iff this trajectory tends to the origin. The worst trajectory is constructed in the following way. We study the locus $Q^{-1}(0)$ (the notation is clarified later) in which the two vector fields Ax and Bx are collinear. We have several cases:

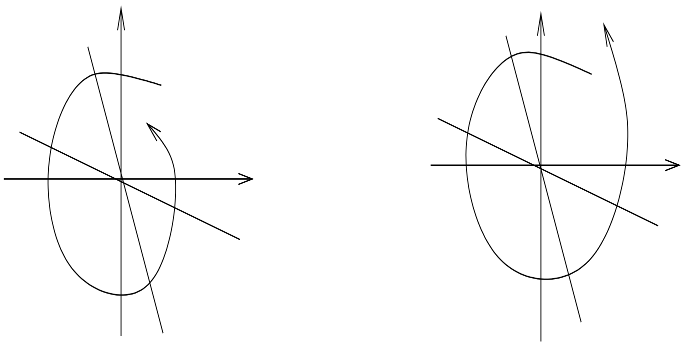
- If $Q^{-1}(0)$ contains only the origin then one vector field points always on the same side of the other and the worst trajectory is a trajectory of the vector field Ax or Bx. In this case the system is asymptotically stable (case **(CC.1)** and **(RC.1)** of Theorem 31).



- If $Q^{-1}(0)$ does not contain only the origin then it is a couple of straight lines passing from the origin (see next session). In this case the worst trajectory is a trajectory that switches, exactly on this locus, from one vector field to the other. If at each point of Q the two vector field have opposite versus then the worst trajectory does not tend to the origin (see figure).



This corresponds to cases **(CC.2.1)** and **(RC.2.1)** of Theorem 31. If at each point of Q the two vector fields have the same versus then the worst trajectory turns around the origin and the system is stable or instable according to the fact that after one turn the distance from the origin is decreasing or increasing.



This corresponds to cases **(CC.2.2)** and **(RC.2.2)** of Theorem 31.

- Finally **(CC.3)** and **(RC.3)** are the degenerate cases in which the two straight lines coincide.

More details are given later.

7.2 Proof of the Stability Theorem

In the following we prove Theorem 31, separately for the two cases in which A and B have both complex or one complex and the other real eigenvalues.

7.2.1 The case in which A and B have both complex eigenvalues

Let $-\delta_A \pm i\omega_A$ ($\delta_A, \omega_A > 0$) be the eigenvalues of A and $-\delta_B \pm i\omega_B$ ($\delta_B, \omega_B > 0$) be the eigenvalues of B. We have $\rho_A = \delta_A/\omega_A, \quad \rho_B = \delta_B/\omega_B$. Chose a system of coordinates in which:

$$A = \begin{pmatrix} -\delta_A & -\omega_A/E \\ \omega_A E & -\delta_A \end{pmatrix} \qquad B = \begin{pmatrix} -\delta_B & -\omega_B \\ \omega_B & -\delta_B \end{pmatrix},$$

where $E \in \mathbb{R} \setminus \{0\}$. Moreover we assume $E \neq \pm 1$ otherwise $[A, B] = 0$. In this case we have $\mathcal{K} = \frac{1}{2}(E + \frac{1}{E})$ and without loss of generality we may assume $|E| > 1$.

The locus in which Ax and Bx are collinear is $Q^{-1}(0)$ where:

$$Q = det(Ax, Bx) = x_1^2(-\delta_A\omega_B + \delta_B\omega_A E) + x_1x_2(\omega_A\omega_B(E - 1/E) + x_2^2(-\delta_A\omega_B + \delta_B\omega_A/E)$$

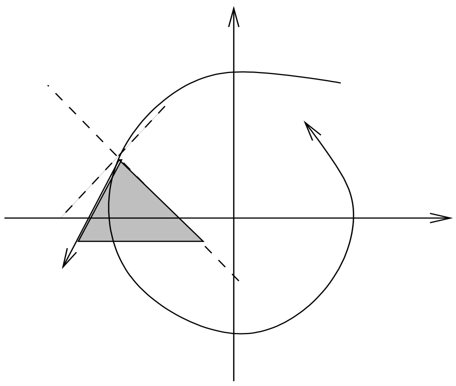
and $x = (x_1, x_2)$. Now let D_{CC} be the discriminant of the quadratic form Q. We have

$$D_{CC} = \omega_A^2\omega_B^2(E - 1/E)^2 - 4(-\delta_A\omega_B + \delta_B\omega_A E)(-\delta_A\omega_B + \delta_B\omega_A/E) = 4\omega_A^2\omega_B^2\mathcal{D}, \quad (7.10)$$

where $\mathcal{D}$ is defined in Definition 35.

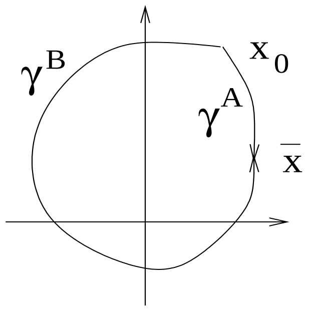
Case 1. If $\mathcal{D} < 0$ then the quadratic form Q has strictly defined sign and $Q^{-1}(0) = \{0\}$. In this case one vector field points always on the same side of the other. Making a suitable change of coordinates and possibly exchanging the label $(_A)$ and $(_B)$ we can realize the situation in which Ax points always on the left of Bx for every $x \in \mathbb{R}^2 \setminus \{0\}$. We have two cases.

- Suppose first $E > 1$. In this case Ax points always in the grey region of the following picture:



Fix an arbitrary measurable switching function $u(.) : [0, \infty[\to [0, 1]$ and let $(x_1(t), x_2(t))$ (resp. $(\rho(t), \theta(t))$ be the Cartesian (resp. polar) coordinates of the solution of $\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t)$, $x(0) = x_0 \in \mathbb{R}^2 \setminus \{0\}$. One can easily conclude that $\dot{\rho}(t) < 0$ for every $t \in [0, +\infty[$ and $(\mathcal{P})$ is true.

- Suppose now $E < -1$. Fix $x_0 \in \mathbb{R}^2 \setminus \{0\}$ and let γ a trajectory of the switched system (7.1) such that $\gamma(0) = x_0$. Let $\gamma^A : [0, t_A] \to \mathbb{R}^2$ (resp. $\gamma^B : [0, t_B] \to \mathbb{R}^2$ a trajectory of the vector field Ax (resp Bx) such that $\gamma^A(0) = x_0$ (resp. $\gamma^B(0) = x_0$) and define t_A and t_B in such a way $\gamma^A(t_A) = \gamma^B(t_B) =: \bar{x}$ is the first intersection point of γ^A and γ^B after x_0.



Let Ω the simply connected closed set whose border is $\partial\Omega = Supp(\gamma^A|_{[0,t_A]} \cup \gamma^B|_{[0,t_B]})$. For every $x \in \partial\Omega$ we have the following. Define $V_u = uAx+(1-u)Bx$. For each $u \in]0, 1[$, V_u points inside Ω Moreover if $x \notin \{x_0, \bar{x}\}$, V_1 (resp. V_0) points inside Ω or it is tangent to $\partial\Omega$. Fix $\bar{t} > \max\{t_A, t_B\}$. we clearly have $\bar{x} := \gamma(\bar{t}) \in \overset{\circ}{\Omega}$. Using homotety invariance of the system (7.1) we may easily conclude that $\lim_{t\to\infty} \gamma(t) = 0$ and $(\mathcal{P})$ is true. This prove case **(CC.1)** of Theorem 31 (see example 1 below).

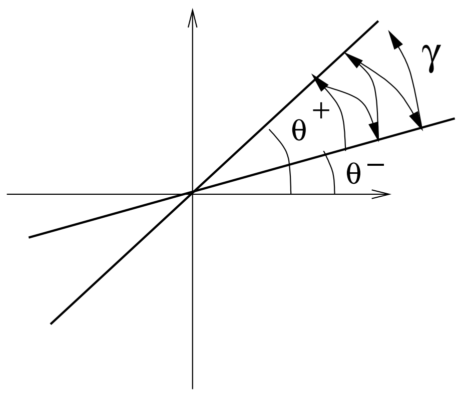
Case 2. If $\mathcal{D} > 0$ then Q has not definite sign and $Q^{-1}(0)$ is a couple of non coinciding straight lines passing form the origin and forming the following angles with the x_1 axis:

$$\theta^{\pm} = \arctan(m^{\pm}), \text{ where:} \tag{7.11}$$

$$m^{\pm} = \frac{-\omega_A\omega_B(E-1/E) \pm \sqrt{D_{CC}}}{2(-\delta_A\omega_B + \delta_B\omega_A/E)} =$$
$$\frac{-(E-1/E) \pm 2\sqrt{\mathcal{D}}}{2(-\rho_A + \rho_B/E)} \quad \text{if } -\rho_A + \rho_B/E \neq 0, \tag{7.12}$$
$$m^- = \infty, \quad m^+ = \frac{\delta_A\omega_B - \delta_B\omega_A E}{\omega_A\omega_B(E-1/E)} =$$
$$\frac{\rho_A - \rho_B E}{E - 1/E} \quad \text{if } -\rho_A + \rho_B/E = 0, \tag{7.13}$$

where we assume $\theta^- \in [-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\theta^+ \in]\theta^-, \theta^- + \pi[$. Notice that if $E < -1$ then $4(-\delta_A\omega_B + \delta_B\omega_A E)(-\delta_A\omega_B + \delta_B\omega_A/E) > 0$ that implies $\theta^\pm \in [-\frac{\pi}{2}, 0[$.

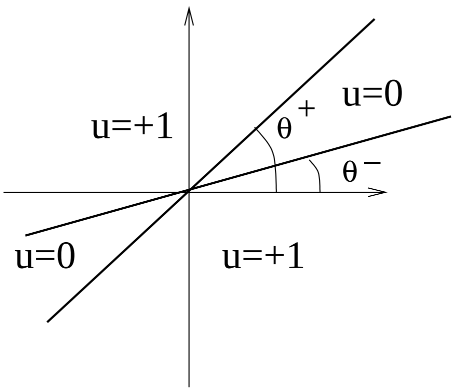
Case 2.1 If $E < -1$ ($\mathcal{K} < -1$) then at each point of $Q^{-1}(0) \setminus \{0\}$, the two vector fields has opposite versus and $(\mathcal{P})$ is false because it is possible to build a couple trajectory switching–function (γ, u) of the system (7.1) such that $l|\gamma(t)| = \infty$ (see figure). Case **(CC.2.1)** of Theorem 31 is proved (see example 4 below).



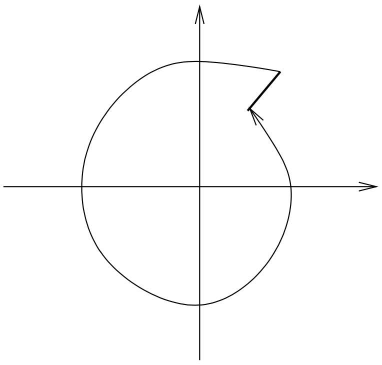
Case 2.2 If $E > 1$ ($\mathcal{K} > 1$) then at each points of $Q^{-1}(0) \setminus \{0\}$ the two vector fields has the same versus (counterclockwise). Fix $x_0 \in \mathbb{R}^2 \setminus \{0\}$ and let $\gamma^M : [0, \infty[\to \mathbb{R}^2$, $\gamma^M(0) = x_0$ be the trajectory corresponding to the feedback:

$$u^M(x) = \begin{cases} 0 \text{ if } \theta \in [\theta^-, \theta^+[\text{ or } \theta \in [\theta^- + \pi, \theta^+ + \pi[\\ +1 \text{ if } \theta \in [\theta^+, \theta^- + \pi[\text{ or } \theta \in [\theta^+ + \pi, \theta^- + 2\pi[, \end{cases} \tag{7.14}$$

where $\theta \in [\theta^-, \theta^- + 2\pi[$ is defined by $x_1 = \rho\cos(\theta)$, $x_2 = \rho\sin(\theta)$



Let $(\rho^M(t), \theta^M(t))$ be the polar coordinates of γ^M and a the time defined by $\theta^M(a) = \theta^M(0) + 2\pi$. If $\rho^M(a) < \rho^M(0)$ then let l be the segment joining the points $(\rho^M(0), \theta^M(0))$ with $(\rho^M(a), \theta^M(a))$, and Ω the simply connected region whose border is $\partial\Omega := Supp(\gamma^M|_{[0,a]} \cup l)$.



For every $x \in \partial\Omega$, we have the following. Define V_u as in Case 1, $E < -1$. For each $u \in]0,1[$, V_u points inside Ω. Moreover if $x \notin \{\gamma^M(0), \gamma^M(a)\}$, V_1 (resp. V_0) point inside Ω or is tangent to $\partial\Omega$. Similarly to Case 1 ($E < 1$) we can conclude $(\mathcal{P})$ true (see example 3 below). On the other side if $\rho^M(a) \geq \rho^M(0)$ then $\gamma^M(t)$ does not tends to the origin and $(\mathcal{P})$ is false (see example 2 below). The condition $\rho^M(a) < \rho^M(0)$ is satisfied iff condition (7.5) holds. Formula (7.5) is obtained in Appendix A . The condition $\rho^M(a) = \rho^M(0)$ (i.e. $\rho_{CC} = 1$) is the case in which we have at least stability for every trajectory of the switching system (7.1). This concludes the proof of case **(CC.2.2)**.

Case 3. If $\mathcal{D} = 0$ then the two straight lines coincide. If $E > 1$ it is easy to understand that we are in the same situation of Case 1. If $E < -1$ then to every $x \in Q$ there exists $u \in [0,1]$ such that $uAx + (1-u)Bx = 0$. In this case $(\mathcal{P})$ is false but we have at least stability for each trajectory of the switched system. This prove case **(CC.3)** of Theorem 31.

EXAMPLES

In the following we give some examples of the various situations in the **(CC)** case. We refer to the following figure.

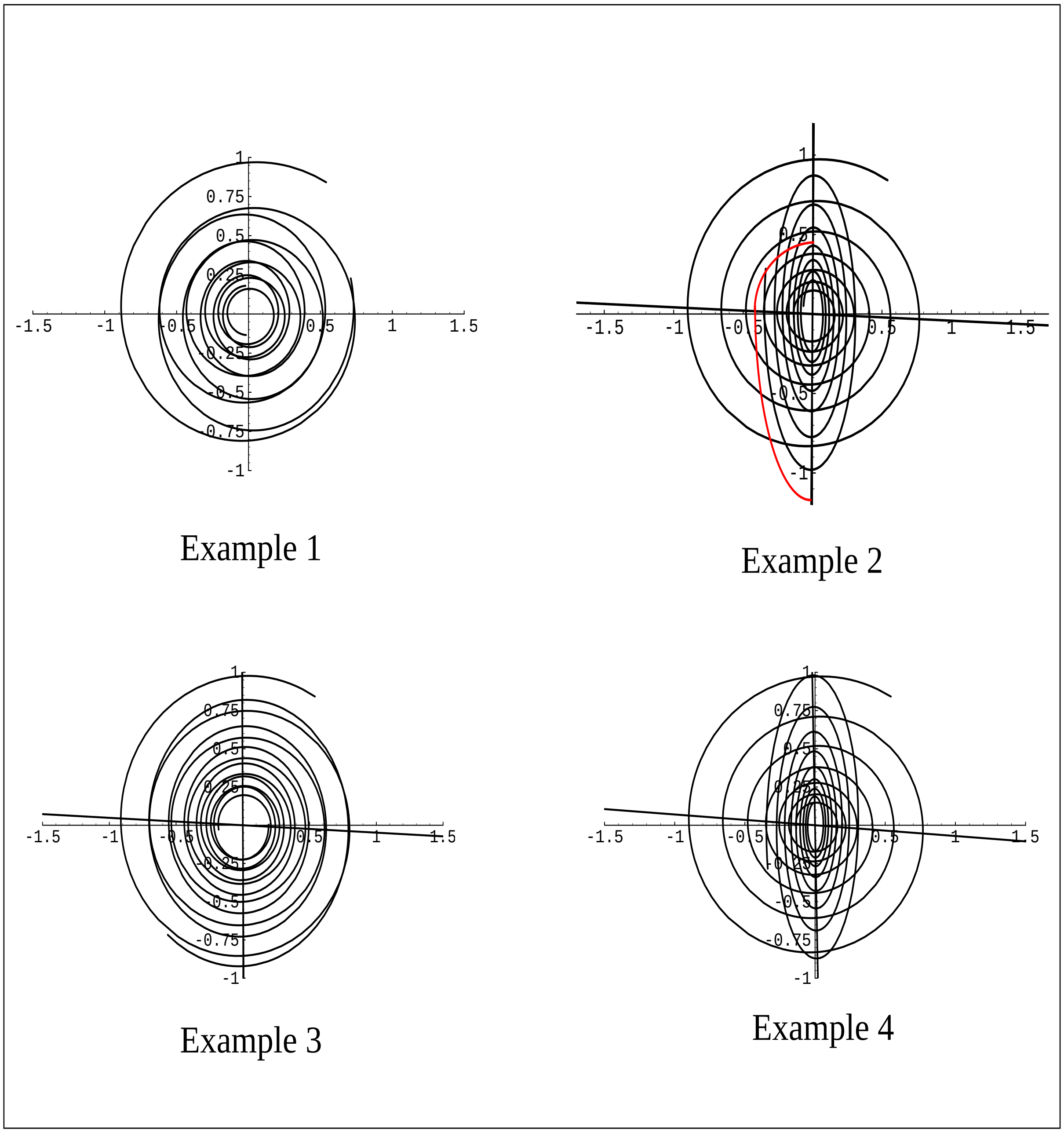


Figure 7.1: Examples in the **(CC)** case

Example 1 $\rho_A = 0.05,\ \rho_B = 0.06,\ \mathcal{K} = -1.005$. In this case $\mathcal{D} \sim -0.002$ and $(\mathcal{P})$ is true. In the figure it is shown two integral curves of the vector fields Ax and Bx. A similar situation but with the two trajectories rotating with the same versus can be obtained with the same values of ρ_A and ρ_B but with $k = +1.00001$. In this case $\mathcal{D} \sim -0.00008$ and $(\mathcal{P})$ is true (cfr. case **(CC.1)**).

Example 2 $\rho_A = 0.0375,\ \rho_B = 0.05,\ \mathcal{K} = 1.67$. In this case $\mathcal{D} \sim 1.79,\ \rho_{CC} \sim 2.62$ and $(\mathcal{P})$ is false. In the figure it is shown two integral curves of the vector fields Ax and Bx, $\mathcal{D}$ that are the two straight lines (one almost coincides with the x_2 axis) and a trajectory γ such that $l|\gamma(t)| = \infty$ (cfr. case **(CC.2.2)**).

Example 3 $\rho_A = 0.0375,\ \rho_B = 0.0425,\ \mathcal{K} = 1.00455$. In this case $\mathcal{D} \sim 0.0091,\ \rho_{CC} \sim 0.96$ and $(\mathcal{P})$ is true (cfr. case **(CC.1)**).

Example 4 Suppose $\rho_A = 0.0375,\ \rho_B = 0.05,\ \mathcal{K} = -1.67$. In this case $\mathcal{D} \sim 1.77$ and $(\mathcal{P})$ is false (cfr. case **(CC.2.1)**).

7.2.2 The case in which A and B have one complex and the other real eigenvalues

Suppose that A has real eigenvalues $\lambda_1,\ \lambda_2$ ($\lambda_1, \lambda_2 < 0$, $|\lambda_2| > |\lambda_1|$) and B complex eigenvalues $\lambda_3 = -\delta + i\omega$, $\lambda_4 = -\delta - i\omega$ ($\delta, \omega > 0$). We have $\rho_A = -i(\lambda_1 + \lambda_2)/(\lambda_1 - \lambda_2)$ and $\rho_B = \delta/\omega$. We recall that $\rho_A/i > 1,\ \rho_B > 0$. Choose a system of coordinates in which:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{7.15}$$

$$\begin{aligned} B &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} := R^{-1}(\varphi)\begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix} R(\varphi) = \\ &= \begin{pmatrix} -\delta - \omega(E - 1/E)\sin(\varphi)\cos(\varphi) & -\omega(E\sin^2(\varphi) + 1/E\cos^2(\varphi)) \\ \omega(E\cos^2(\varphi) + 1/E\sin^2(\varphi)) & -\delta + \omega(E - 1/E)\sin(\varphi)\cos(\varphi) \end{pmatrix}, \end{aligned} \tag{7.16}$$

where $R(\varphi)$ is defined as in formula (7.4). We have $\mathcal{K} = i(E - 1/E)\cos(\varphi)\sin(\varphi) \in i\mathbb{R}$ and without loss of generality we may assume $\varphi \in [0, \pi/2[$, $|E| \geq 1$.

Similarly to the previous sub–section, the locus in which Ax and Bx are collinear is $Q^{-1}(0)$ where:

$$Q = det(Ax, Bx) = x_1^2(\lambda_1 c) + x_1 x_2 \bar{\chi} + x_2^2(-\lambda_2 b),$$

and by definition $\bar{\chi} := \lambda_1 d - \lambda_2 a = (\lambda_1 + \lambda_2)\omega\mathcal{K}/i - (\lambda_1 - \lambda_2)\delta = (\lambda_1 - \lambda_2)\omega\chi$, where $\chi := \rho_A\mathcal{K} - \rho_B$ (see Theorem 31). In this case the discriminant of the quadratic form Q is:

$$D_{RC} = \bar{\chi}^2 + 4\lambda_1\lambda_2 bc = \bar{\chi}^2 - 4\lambda_1\lambda_2\omega^2(-\mathcal{K}^2 + 1) = -\omega^2(\lambda_1 - \lambda_2)^2\mathcal{D}. \tag{7.17}$$

Notice that $\chi = 0$ implies $\bar{\chi} = 0$, that implies $D_{RC} < 0$ i.e $\mathcal{D} > 0$. Moreover $\chi > 0$ implies $\mathcal{K}/i < 0$ that implies $E < -1$. Similarly to the previous subsection we have the following cases

Case 1. If $D_{RC} < 0$ ($\mathcal{D} > 0$) then ($\mathcal{P}$) is true (see example 1 below).

Case 2. If $D_{RC} > 0$ ($\mathcal{D} < 0$) then $Q^{-1}(0)$ is a couple of non coinciding straight lines passing from the origin and forming the following angles with the x_1 axis:

$$\begin{aligned} &\theta^{\pm} = \arctan(m^{\pm}), \\ &m^{\pm} := \frac{-\bar{\chi} \pm \sqrt{D_{RC}}}{2(-\lambda_2 b)} = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{2\frac{\lambda_2}{\lambda_1 - \lambda_2}(E\sin^2(\varphi) + 1/E\cos^2(\varphi))}. \end{aligned} \tag{7.18}$$

From (7.17) it follows $D_{RC} < \bar{\chi}^2$ (i.e. $-\mathcal{D} < \chi^2$), so that in this case we have $\chi \neq 0$ and we may assume:

$$\begin{cases} \theta^-, \theta^+ \in]0, \pi/2[\text{ if } \chi \text{ and } E \text{ have the same sign,} \\ \theta^-, \theta^+ \in]-\pi/2, 0[\text{ if } \chi \text{ and } E \text{ have opposite sign.} \end{cases}$$

Case 2.1 If $\chi > 0$ then $\mathcal{K}/i < 0$ that implies $E < -1$ and we have $\theta^- \theta^+ \in]-\pi/2, 0[$. In this case at each point of $Q^{-1}(0) \setminus \{0\}$ the two vector fields have opposite versus. The same argument of case 2.1 of Section 7.2.1 shows that ($\mathcal{P}$) is false (see example 4 below).

Case 2.2 If $\chi < 0$ then in both cases $E \geq 1,\ E \leq -1$ at each point of $Q^{-1}(0) \setminus \{0\}$ the two vector fields have the same versus.

Case 2.2.A If $E \leq -1$ (that implies $\mathcal{K}/i \leq 0$) then ($\mathcal{P}$) is true because of the following argument.

From $\chi = \rho_A \mathcal{K} - \rho_B < 0$ we have:

$$-\frac{\mathcal{K}/i}{\rho_B} < \frac{1}{\rho_A/i} < 1. \tag{7.19}$$

Now let γ an integral of the vector field Bx and $(\rho(t), \theta(t))$ its polar coordinates. We have:

$$\gamma(t) = R(\varphi) \begin{pmatrix} \rho_0 e^{-\delta t} \cos(\omega t + \varphi_0) \\ \rho_0 E e^{-\delta t} \sin(\omega t + \varphi_0), \end{pmatrix}$$

and $\rho(t) = \rho_0 e^{-\delta t}\sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)}$. Now we prove that the condition (7.19) implies $\dot{\rho}(t) \leq 0$ for every $t \in Dom(\gamma)$ that clearly implies ($\mathcal{P}$) true. We have:

$$\begin{aligned} \dot{\rho}(t) \quad = \quad & \rho_0 e^{-\delta t} \left(\frac{(E^2 - 1)\omega \sin(\omega t + \varphi_0)\cos(\omega t + \varphi_0)}{\sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)}} - \right. \\ & \left. \delta \sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)} \right). \end{aligned}$$

Therefore $\dot{\rho}(t) < 0$ iff

$$\frac{(E^2-1)\omega\sin(\omega t+\varphi_0)\cos(\omega t+\varphi_0)}{\sqrt{\cos^2(\omega t+\varphi_0)+E^2\sin^2(\omega t+\varphi_0)}} - \delta\sqrt{\cos^2(\omega t+\varphi_0)+E^2\sin^2(\omega t+\varphi_0)} < 0,$$

or equivalently iff:

$$\cos^2(\omega t+\varphi_0)+E^2\sin^2(\omega t+\varphi_0) - (E^2-1)\frac{\omega}{\delta}\sin(\omega t+\varphi_0)\cos(\omega t+\varphi_0) > 0. \tag{7.20}$$

Now if $(E^2-1)\frac{\omega}{\delta} \leq -2E$, that choosing a system of coordinates in which $\varphi = \pi/4$ it is equivalent to $\frac{-\mathcal{K}/i}{\rho_B} \leq 1$ (see Appendix B), the condition (7.20) is satisfied. Hence from (7.19) we can conclude $\dot{\rho}(t) < 0$ for each $t \in Dom(\gamma)$ and $(\mathcal{P})$ is true (see example 5 below).

Case 2.2.B If $E \geq 1$ (that implies $\mathcal{K}/i \geq 0$) the $(\mathcal{P})$ is true iff condition (7.6) is satisfied (see Appendix B). Notice that in the case $\mathcal{K} = 0$ we clearly have $\rho_{RC} < 1$ and $(\mathcal{P})$ is true (see examples 2 and 3 below). The case in which $\rho_{RC} = 1$ is the case in which we have at least stability.

Case 3. If $\mathcal{D} = 0$ then the two straight lines coincide. If $\chi < 0$ it is easy to understand that we are in the same situation of Case 1. If $\chi > 0$ then to every $x \in Q^{-1}(0)$ there exists $u \in [0,1]$ such that $uAx + (1-u)Bx = 0$. In this case $(\mathcal{P})$ is false but we have at least stability for each trajectory of the switched system. This prove case **(RC.3)** of Theorem 31.

This concludes the proof of cases **(RC)**.

EXAMPLES

In the following we give some examples of the various situations in the **(RC)** case. We refer to the following figure.

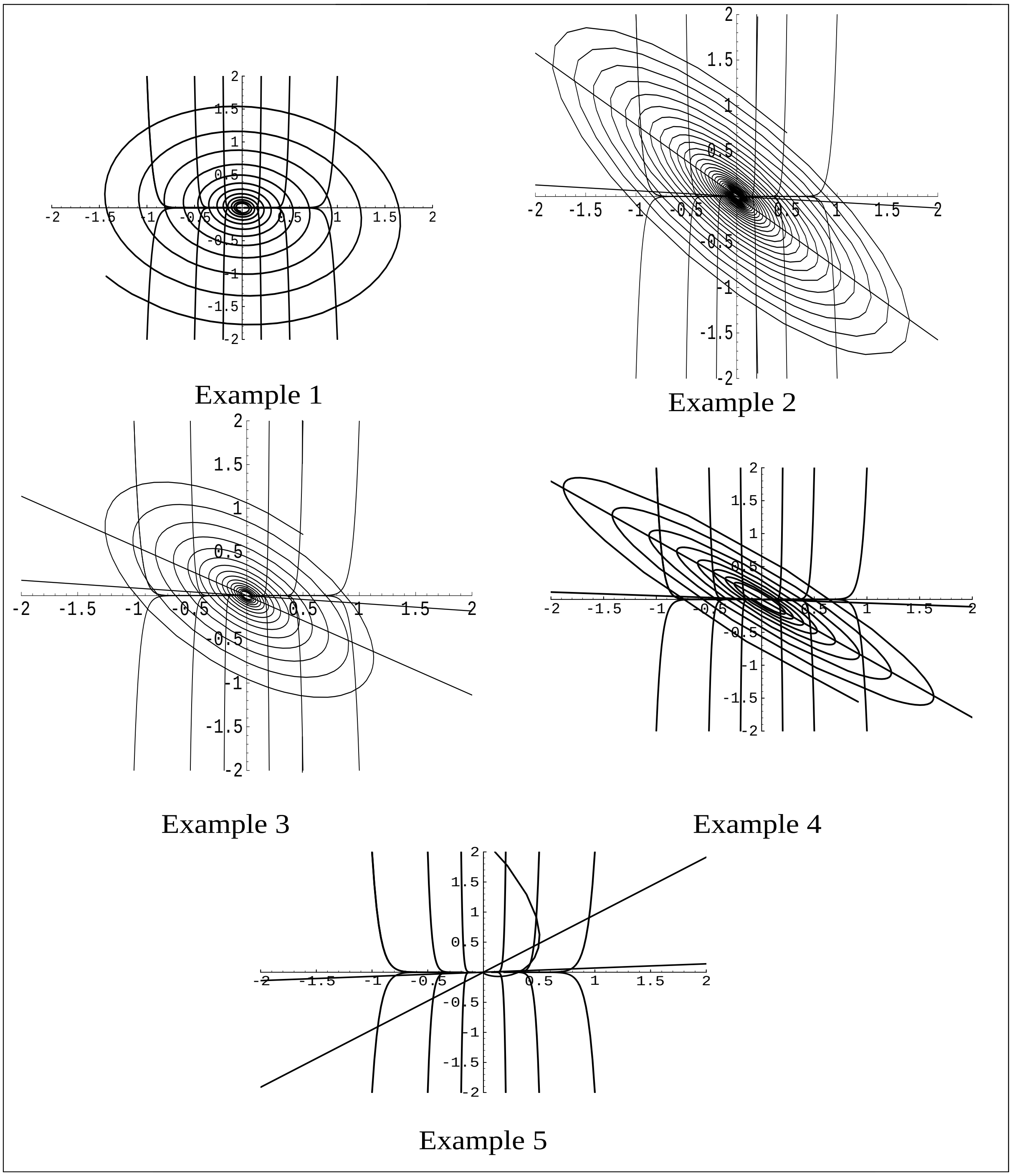


Figure 7.2: Examples in the **(RC)** case

Example 1 $\rho_A/i = 1.11$, $\rho_B = 0.045$, $\mathcal{K}/i = 0.095$. In this case $\chi \sim -0.15$, $\mathcal{D} \sim 0.2$ and $(\mathcal{P})$ is true (cfr. case **(RC.1)**)

Example 2 $\rho_A/i = 1.11$, $\rho_B = 0.02$, $\mathcal{K}/i = 1.33$. In this case $\chi \sim -1.49$, $\mathcal{D} \sim -1.62$, $\rho_{RC} \sim 1.4$ and $(\mathcal{P})$ is false (cfr. case **(RC.2.2.B)**).

Example 3 $\rho_A/i = 1.11$, $\rho_B = 0.03$, $\mathcal{K}/i = 0.75$. In this case $\chi \sim -0.85$, $\mathcal{D} \sim -0.37$, $\rho_{RC} \sim 0.98$ and $(\mathcal{P})$ is true (cfr. case **(RC.2.2.B)**).

Example 4 $\rho_A/i = 1.11$, $\rho_B = 0.045$, $\mathcal{K}/i = -2.4$. In this case $\chi \sim 2.6$, $\mathcal{D} \sim -5.3$ and $(\mathcal{P})$ is false (cfr. case **(RC.2.1)**).

Example 5 $\rho_A/i = 1.14$, $\rho_B = 1.67$, $\mathcal{K}/i = -0.42$. In this case $\chi \sim -1.19$, $\mathcal{D} \sim -1.06$ and $(\mathcal{P})$ is true (cfr. case **(RC.2.2.A)**).

7.3 Study of the Stability Domain

7.3.1 Stability domain in the Complex–Complex case

In Figure 7.3 it is shown the asymptotic stability domain in the (ρ_A, ρ_B) plane for a fixed value of $\mathcal{K}$ in the case in which both A and B have complex eigenvalues.

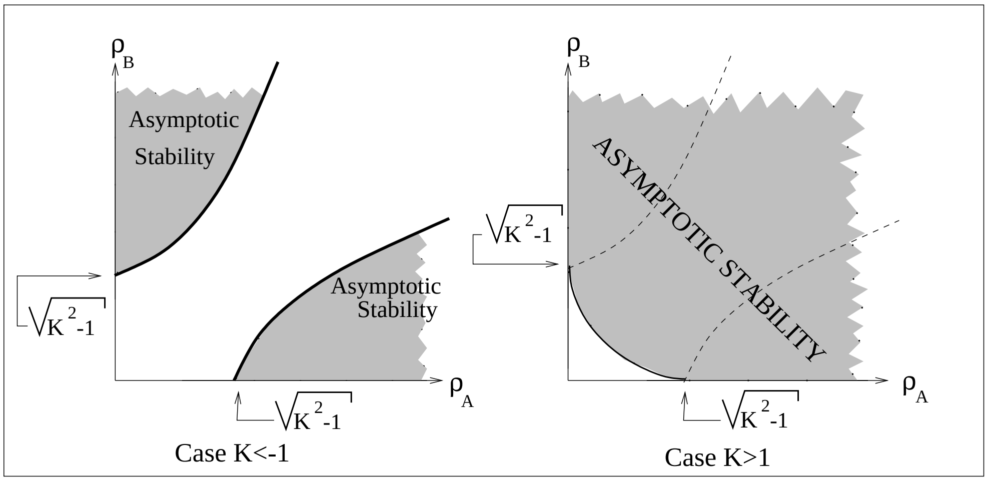


Figure 7.3: The asymptotic stability domain (the grey region) for a fixed value of $\mathcal{K}$, in the Complex–Complex case.

In the case $\mathcal{K} < -1$ the stability domain is determined by the condition $\mathcal{D} < 0$. The set of value of ρ_A and ρ_B such that $\mathcal{D} = 0$ are the two curved lines of Figure 7.3 (case $\mathcal{K} < -1$) of equations $\rho_B = \rho_A \mathcal{K} \pm \sqrt{(\rho_A^2 + 1)(\mathcal{K}^2 - 1)}$. The points of intersection with the two axis are:

$$(\rho_A, \rho_B) = (\sqrt{\mathcal{K}^2 - 1}, 0) \tag{7.21}$$

$$(\rho_A, \rho_B) = (0, \sqrt{\mathcal{K}^2 - 1}) \tag{7.22}$$

In this case, for fixed values of $\mathcal{K}$ the stability domain is constituted by two connected open convex unbounded regions, while we have at least stability on this set plus the points in which $\mathcal{D} = 0$

In the case $\mathcal{K} > 1$ the asymptotic stability domain is determined by the condition $\rho_{CC} < 1$. In Figure 7.3 (case $\mathcal{K} > 1$) the locus $\mathcal{D} = 0$ is drawn with dotted lines while the locus $\rho_{CC} = 1$ with a solid line. The points in which the two loci intersect each other and intersect the two axis are given again by formulas (7.21) and (7.22). To study the convexity of the asymptotic stability domain in this case we have to check if expressing the locus $\rho_{CC} = 0$ as $\rho_B = f_{\mathcal{K}}(\rho_A)$, we find a convex function. In the following the label $(_{\mathcal{K}})$ is a parameter and it will be dropped. Let us indicate the derivative of ρ_{CC} with respect to the first and second variable as $\left(\rho_{CC}\right)_1$ and $\left(\rho_{CC}\right)_2$. We have that:

$$f'(\rho_A) = F\left(\rho_A, f(\rho_A)\right) \text{ where } F(\rho_A, \rho_B) := -\frac{\left(\rho_{CC}\right)_1}{\left(\rho_{CC}\right)_2} = -\frac{\arctan(\frac{\rho_B - \rho_A \mathcal{K}}{\sqrt{\mathcal{D}}}) + \frac{\pi}{2}}{\arctan(\frac{\rho_A - \rho_B \mathcal{K}}{\sqrt{\mathcal{D}}}) + \frac{\pi}{2}}$$

$$\begin{aligned}
&f''(\rho_A) = G\left(\rho_A, f(\rho_A)\right) \text{ where:}\\
&G(\rho_A, \rho_B) = \frac{\partial F(\rho_A, \rho_B)}{\partial \rho_A} + \frac{\partial F(\rho_A, \rho_B)}{\partial \rho_B} F(\rho_A, \rho_B) =\\
&\frac{2}{(1+\rho_A^2)\ (1+\rho_B^2)\ \sqrt{\mathcal{D}} \left(\pi + 2 \arctan(\frac{\rho_A - \rho_B \mathcal{K}}{\sqrt{\mathcal{D}}})\right)^3} \times\\
&\Big([\rho_A^3\,\rho_B + \rho_A\,\rho_B^3 + 2(1+\rho_A^2+\rho_B^2+\rho_A\rho_B+\rho_A^2\rho_B^2) + \mathcal{K}(2+\rho_A^2+\rho_B^2)]\,\pi^2 +\\
&4\left(1+\rho_B^2\right)\left(1+\rho_A^2+\rho_A\,\rho_B+\mathcal{K}\right)\,\pi\,\arctan(\frac{\rho_A - \rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}}) +\\
&4\left(1+\rho_A^2\right)\left(1+\rho_B^2+\rho_B\,\rho_A+\mathcal{K}\right)\,\pi\,\arctan(\frac{\rho_B - \rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}}) +\\
&4\left(1+\rho_A^2\right)\left(\rho_A\,\rho_B+\mathcal{K}\right)\,\arctan(\frac{\rho_B - \rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}})^2 +\\
&4\left(1+\rho_B^2\right)\left(\rho_A\,\rho_B+\mathcal{K}\right)\,\arctan(\frac{\rho_A - \rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}})^2 +\\
&8\left(1+\rho_A^2\right)\left(1+\rho_B^2\right)\,\arctan(\frac{\rho_B - \rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}})\,\arctan(\frac{\rho_A - \rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}})\Big)
\end{aligned}$$

Now the only terms that can be negative are the ones in the 3th and 4th rows, but it easy to check numerically that the sum of these two terms with the one in the second row is always bigger than zero. It follows the convexity. In this case the asymptotic stability domain is a convex open unbounded region, while the stability domain is a convex not open unbounded region (we have to add the points such that $\rho_{CC} = 1$).

7.3.2 Stability domain in the Real–Complex case

In the case in which A and B have one complex and the other real eigenvalues, the asymptotic stability domain is drawn in figure 7.4. We recall that $\rho_A/i > 1$, $\rho_B > 0$,

$\mathcal{K}/i \in \mathbb{R}$.

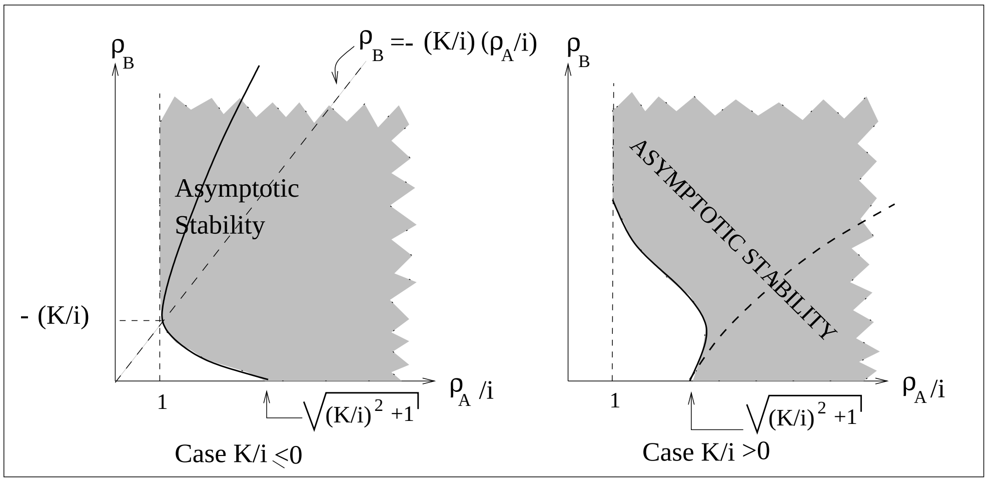


Figure 7.4: The asymptotic stability domain (the grey region) for a fixed value of $\mathcal{K}$, in the Real–Complex case.

In the case $\chi > 0$ (that implies $\mathcal{K}/i < 0$ and $\rho_B < (-\mathcal{K}/i)(\rho_A/i)$), the asymptotic stability domain is determined by the condition $\mathcal{D} > 0$. The locus $\mathcal{D} = 0$ is the set of points such that: $\rho_B = -(\rho_A/i)(\mathcal{K}/i) \pm \sqrt{(-(\rho_A/i)^2 + 1)(-(\mathcal{K}/i)^2 - 1)}$. The intersection point with the ρ_A axis is:

$$(\rho_A/i, \rho_B) = (\sqrt{(\mathcal{K}/i)^2 + 1}, 0), \tag{7.23}$$

and the intersection with the $\rho_A/i = 1$ set is:

$$(\rho_A/i, \rho_B) = (1, -(\mathcal{K}/i)).$$

In the case $\chi < 0$ and $\mathcal{K}/i \leq 0$ we have asymptotic stability. We conclude that in the case $\mathcal{K}/i \leq 0$, the asymptotic stability domain is a convex open unbounded region (see Figure 7.4 (case $\mathcal{K}/i \leq 0$)) while to have at least stability we have to add the points in which $\mathcal{D} = 0$.

In the case $\chi < 0$ and $\mathcal{K}/i > 0$ the asymptotic stability domain is determined by the condition $\rho_{RC} < 1$. In Figure 7.4 (case $\mathcal{K}/i > 0$) the locus $\mathcal{D} = 0$ is drawn with dotted lines while the locus $\rho_{CC} = 1$ with a solid line. The points in which the two loci intersect each other are given by formula (7.23). In this case the asymptotic stability domain is a non convex open unbounded region. Again the points in which we have at least stability are the points in which we have asymptotic stability plus the points such that $\rho_{RC} = 1$.

7.4 Final Remarks

Using the results of [1, 10, 27] and by Theorem 31 we have a complete algorithm to study the asymptotic stability of a switched linear system in any dimension at least in the case in which:

$$A_u = uA^1x + (1-u)A^2x, \qquad u \in [0,1], \quad A^1, A^2 \in \mathbb{R}^{n\times n}$$

where A and B are diagonalizable. The case in which A or B are not diagonalizable can be treated with the same techniques. Generalization can be done for more complex sets U. One is the following m–input system:

$$\dot{x} = \sum_{i=1}^{m} u_i A^i x, \quad \sum_{i=1}^{m} u_i = 1, \quad u_i \geq 0 \ \ (i = 1, ..., m), \quad x \in \mathbb{R}^n, \quad A^1, ..., A^m \in \mathbb{R}^{n\times n}.$$

With exactly the same techniques used in this paper, one can study the stabilizability of a control system of the kind:

$$\dot{x} = A_u x, \quad x \in \mathbb{R}^n, \quad A_u \in \mathbb{R}^{n\times n}, \quad u \in U,$$

where all the matrices have eigenvalues with strictly *positive* real part. This can be useful for applications.

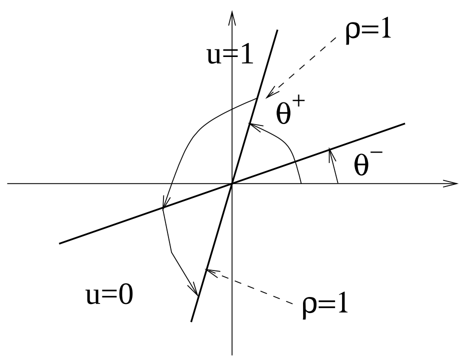
Some results can be obtained for the non-linear version of the problem treated in this paper:

$$\dot{x} = uF(x) + (1-u)G(x), \tag{7.24}$$

where $x \in \mathbb{R}^2$, $F(.)$, $G(.)$ are $\mathcal{C}^\infty$ generic functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $F(0) = 0$, $G(0) = 0$ and the two dynamical systems $\dot{x} = F(x)$, $\dot{x} = G(x)$ are globally asymptotically stable at the origin. We are interested to study under which conditions on $F(.)$ and $G(.)$ the origin of the system (7.24) is globally asymptotically stable for every measurable function $u(.) : [0, \infty[\to [0,1]$. We study all these generalizations in [10].

Appendix A: Proof of the formula (7.5)

We refer to the following figure.



Let $\rho(t),\ \theta(t)$ (resp. $x(t), y(t)$) the polar coordinates (resp. Cartesian) of $\gamma^M(t)$ where we fix the initial condition by setting $\rho(0)=1,\ \theta(0)=\theta^+$. We have to check if at the time a such that $\theta(a)=\theta(0)+2\pi$ we have $\rho(a)<1$. Due to the symmetries of the system this happen iff at the time $\bar{t}$ such that $\theta(\bar{t})=\theta^+ + \pi$, we have $\rho_{RC} := \rho(\bar{t}) < 1$. Notice that $\bar{t}=a/2$. The trajectory $\gamma^M(t)$ corresponds to the constant switching function $u=+1$ up to the time t' in which $\theta(t')=\theta^- + \pi$. This time is defined by the equations:

$$
\begin{aligned}
&x(t') = \rho_0 e^{-\delta_A t'}\cos(\omega_A t' + \theta_E^+),\\
&y(t') = \rho_0 E e^{-\delta_A t'}\sin(\omega_A t' + \theta_E^+),\\
&\theta_E^+ = \arctan\left(\frac{m^+}{E}\right) \in \left\{ \begin{array}{l} [-\pi/2, \pi/2[\text{ if } \theta^+ \in [-\pi/2,\pi/2[\\ ]\pi/2, 3\pi/4[\text{ if } \theta^+ \in]\pi/2, 3\pi/4[\end{array} \right. ,\\
&\rho_0 = (\cos^2(\theta_E^+) + E^2\sin^2(\theta_E^+))^{-1/2}\\
&y(t') = m^- x(t').
\end{aligned}
$$

It follows $\tan(\omega_A t' + \theta_E^+) = m^-/E$. If we set $\theta_E^- = \arctan(m^-/E) \in]\theta_E^+, \theta_E^+ + \pi[$ we have $t' = (\theta_E^- - \theta_E^+)/\omega_A$.

After time t', $\gamma^M_(t)$ corresponds to the constant switching function $u=0$ up to the first time $\bar{t}$ in which $\theta(\bar{t}) = \theta^+ + \pi$. This time is defined by the equations:

$$
\begin{aligned}
&x(\bar{t}) = \rho(t')e^{-\delta_B(\bar{t}-t')}\cos(\omega_B(\bar{t}-t') + \theta^- + \pi),\\
&y(\bar{t}) = \rho(t')e^{-\delta_B(\bar{t}-t')}\sin(\omega_B(\bar{t}-t') + \theta^- + \pi),\\
&\rho(t') = \rho_0 e^{-\frac{\delta_A}{\omega_A}(\theta_E^- - \theta_E^+)}\sqrt{\cos^2(\theta_E^-) + E^2\sin^2(\theta_E^-)},\\
&y(\bar{t}) = m^+ x(\bar{t}).
\end{aligned}
$$

It follows $\tan(\omega_B(\bar{t}-t') + \theta^- + \pi) = \tan(\omega_B(\bar{t}-t') + \theta^-) = m^+ = \tan(\theta^+)$ and we have $\bar{t} = (\theta^+ - \theta^-)/\omega_B + t'$. Finally:

$$
\bar{\rho} = \rho(\bar{t}) = \rho(t')e^{-\frac{\delta_B}{\omega_B}(\bar{t}-t')} = e^{-\frac{\delta_A}{\omega_A}(\theta_E^- - \theta_E^+) - \frac{\delta_B}{\omega_B}(\theta^+ - \theta^-)}\sqrt{\frac{\cos^2\theta_E^- + E^2\sin^2(\theta_E^-)}{\cos^2\theta_E^+ + E^2\sin^2(\theta_E^+)}}.
$$

This formula is not in a good form because it is not explicitly invariant for the exchange of δ_A, ω_A with δ_B, ω_B, and because the quantity E does not appear only in the form $E+1/E$. Recalling the definition of $\rho_A, \rho_B, \mathcal{K}$ (see Definition 35) and using the equality:

$$
\arctan a - \arctan b = \arctan\left(\frac{ab+1}{b-a} + \pi/2\right)
$$

that holds for $a > b$ it is possible to obtain the relation:

$$
\begin{aligned}
-\frac{\delta_A}{\omega_A}(\theta_E^- - \theta_E^+) &= -\rho_A\left(\arctan\left(\frac{-\rho_A\mathcal{K} + \rho_B}{\sqrt{\mathcal{D}}}\right) + \pi/2\right)\\
-\frac{\delta_B}{\omega_B}(\theta^+ - \theta^-) &= -\rho_B\left(\arctan\left(\frac{\rho_A - \rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right) + \pi/2\right).
\end{aligned}
$$

Moreover with elementary computation we can show that:

$$\sqrt{\frac{\cos^2\theta_E^- + E^2\sin^2(\theta_E^-)}{\cos^2\theta_E^+ + E^2\sin^2(\theta_E^+)}} = \sqrt{\frac{\rho_A\rho_B + \sqrt{\mathcal{D}}}{\rho_A\rho_B - \sqrt{\mathcal{D}}}}.$$

Formula (7.5) is obtained.

Appendix B: Proof of the formula (7.6)

To obtain a result that explicitly do not depends on the choice of the system of coordinates, we need to write the formulas of Section 7.2.2 in a more invariant way. Set:

$$\psi = \sqrt{\frac{E\cos^2\varphi + 1/E\sin^2\varphi}{E\sin^2\varphi + 1/E\cos^2\varphi}}$$

and make the coordinates transformation:

$$x \to \Psi(\psi)x, \quad \text{where } \Psi(\psi) := \begin{pmatrix} 1 & 0 \\ 0 & \psi \end{pmatrix}.$$

In this case ($E \geq 1$) the new coordinates A, B and $\theta^\pm$ have the expressions:

$$A = \Psi^{-1}(\psi)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\Psi(\psi) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$B = \Psi^{-1}(\psi)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Psi(\psi) = \begin{pmatrix} -\delta - \omega\mathcal{K}/i & -\omega\sqrt{-\mathcal{K}^2+1} \\ +\omega\sqrt{-\mathcal{K}^2+1} & -\delta + \omega\mathcal{K}/i \end{pmatrix},$$

$$\theta^\pm = \arctan m^\pm, \quad m^\pm = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{2\frac{\lambda_2}{\lambda_1-\lambda_2}\sqrt{-\mathcal{K}^2+1}} = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\sqrt{-\mathcal{K}^2+1}}.$$

Equivalently, we can use the expressions (7.15), (7.16), (7.18) for $A, B, \theta^\pm$ with $E \geq 1$ and $\varphi = \pi/4$.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{7.25}$$

$$B = R^{-1}(\pi/4)\begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix}R(\pi/4) \tag{7.26}$$

$$\theta^\pm = \arctan m^\pm, \quad m^\pm := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{\frac{\lambda_2}{\lambda_1-\lambda_2}(E+1/E)} = 2\frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)(E+1/E)}. \tag{7.27}$$

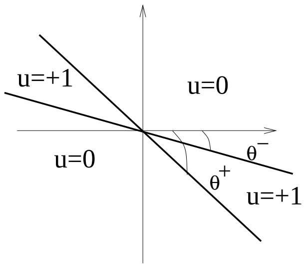
The relation between $\mathcal{K}$ and E are:

$$\mathcal{K} = i\frac{1}{2}(E - 1/E), \quad E = \mathcal{K}/i + \sqrt{-\mathcal{K}^2+1}.$$

Moreover we are considering the case $\chi < 0$ so that $\theta^+, \theta^- \in]-\pi/2, 0[$. From (7.27) it follows $\theta^+ < \theta^-$.

In this case $\gamma^M(.)$ corresponds to the feedback (see figure):

$$u(x) = \begin{cases} 1 \text{ if } \theta \in]\theta^+, \theta^-[\text{ or } \theta \in]\theta^+ + \pi, \theta^- + \pi[\\ 0 \text{ if } \theta \in]\theta^-, \theta^+ + \pi[\text{ or } \theta \in]\theta^- + \pi, \theta^+ + 2\pi[\end{cases}$$



Make the following coordinates transformation: $x \to \bar{x} = R(\pi/4)x$. We have:

$$A \to \bar{A} = R(\pi/4) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R^{-1}(\pi/4)$$

$$B \to \bar{B} = \begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix}$$

$$\theta^\pm \to \bar{\theta}^\pm = \theta^\pm - \pi/4 = \arctan \bar{m}^\pm \in [3/4\pi, \pi/4[, \quad \bar{m}^\pm := \frac{m^\pm - 1}{m^\pm + 1}.$$

Similarly to Appendix A, we compute γ^M in polar coordinates with the initial condition $\rho(0) = 1,\ \theta(0) = \bar{\theta}^-$. Let t' the first time such that $\theta(t') = \bar{\theta}^+ + \pi$. we have:

$$t' = (\xi^+ - \xi^-)/\omega$$

$$\rho(t') = e^{-\frac{\delta}{\omega}(\xi^+ - \xi^-)} \sqrt{\frac{\cos^2 \xi^+ + E^2 \sin^2 \xi^+}{\cos^2 \xi^- + E^2 \sin^2 \xi^-}},$$

where $\xi^\pm := \arctan(\bar{m}^\pm/E), \quad \xi^+ \in]\xi^-, \xi^- + \pi[$.

Now we come back to the old coordinates ($\bar{x} \to x = R^{-1}(\pi/4)\bar{x}$) and we integrate Bx up to the first time $\bar{t}$ such that $\theta(\bar{t}) = \theta^- + \pi$. We have:

$$x(\bar{t}) = \rho(t') \cos(\theta^+ + \pi) e^{\lambda_1(\bar{t} - t')}$$

$$y(\bar{t}) = \rho(t') \sin(\theta^+ + \pi) e^{\lambda_2(\bar{t} - t')}$$

$$y(\bar{t}) = m^- x(\bar{t})$$

It follows:

$$m^+ e^{((\lambda_2 - \lambda_1)(\bar{t} - t'))} = n^- \Longrightarrow \bar{t} - t' = \frac{1}{\lambda_2 - \lambda_1} \ln \left(\frac{m^-}{m^+} \right).$$

Finally

$$
\begin{aligned}
\rho_{RC} \;&:=\; \rho(\bar{t}) = \rho(t')\sqrt{\cos^2\theta^+ e^{\frac{\lambda_1}{\lambda_2-\lambda_1}\ln(m^-/m^+)} + \sin^2\theta^+ e^{\frac{\lambda_2}{\lambda_2-\lambda_1}\ln(m^-/m^+)}} = \\
&= e^{-\frac{\delta}{\omega}(\xi^+-\xi^-)}\sqrt{\frac{\cos^2\xi^+ + E^2\sin^2\xi^+}{\cos^2\xi^- + E^2\sin^2\xi^-}}\sqrt{\cos^2\theta^+\left(\frac{m^-}{m^+}\right)^{\frac{\lambda_1}{\lambda_2-\lambda_1}} + \sin^2\theta^+\left(\frac{m^-}{m^+}\right)^{\frac{\lambda_2}{\lambda_2-\lambda_1}}} \\
&= e^{-\rho_B(\xi^+-\xi^-)}\sqrt{\frac{\cos^2\xi^+ + E^2\sin^2\xi^+}{\cos^2\xi^- + E^2\sin^2\xi^-}}\times \\
&\quad \sqrt{\cos^2\theta^+\left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i+1)} + \sin^2\theta^+\left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i-1)}},
\end{aligned}
$$

that is formula (7.6). This formula is complicated but acceptable because there are no further symmetries.

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