

Dynamical Systems: Introduction & Definition

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1 Introduction

Given the problem to study a system we have to find a way to express its *state*. Usually we can use a set of variables, like a vector x , so that knowing this vector we can (in theory) predict the future states of the system. For example, in mechanics problems, we have to know positions and speeds.

If the system is not in equilibrium, it will change as time flows, and so hopefully will change the describing vector of variables too.

We may call the initial vector $x(t_0)$, to remember us that is related to a time t_0 . Now after some time Δt things have changed and we have another vector. We may call it $x(t_0 + \Delta t)$. These vectors are both taken from a vectorial space X where each element represents a possible configuration, so $x(t_0) \in X$ and $x(t_0 + \Delta t) \in X$. We'd like to find a rule between $x(t_0)$ and $x(t_0 + \Delta t)$. It comes out that it may be expressed in different ways.

Usually we know how the system is going to change when it is in a certain state. But the state is expressed by the vector $x(t)$ and its change rate is just its differential versus time, $\frac{dx}{dt}(t)$. So we found ourselves managing a *differential equation* like the following

$$\frac{dx}{dt}(t) = f(x(t)) \quad (1)$$

Here we use the fact that $x(t)$ contains all the informations about the state of the systems, so we don't need to make f depend on time t explicitly, neither from older states ($t' < t$). By this I mean that there are no forces that are time-dependent, so that if you have two times t_1 and t_2 , $t_1 \neq t_2$, such that

$$x(t_1) = x(t_2) \quad (2)$$

then for all $\Delta t > 0$ we have¹

$$x(t_1 + \Delta t) = x(t_2 + \Delta t) \quad (3)$$

¹This is not true for $\Delta t < 0$.

If this is not possible, then we can't describe the system with (1).

Often it's not possible to analytically solve equation (1), so we can try a numerical approach, but supposing that we are able to integrate $f(x(t))$ respect t , and that the indefinite integral is $F(t)$, we have²

$$\begin{aligned} \int_{t_0}^{t_0+\Delta t} \frac{dx}{d\tau}(\tau)d\tau &= \int_{t_0}^{t_0+\Delta t} f(x(\tau))d\tau \\ x(t_0 + \Delta t) - x(t) &= F(t_0 + \Delta t) - F(t_0) \end{aligned} \quad (4)$$

Now we'd like to invert the function $x(t)$, but it may not be globally invertible, for example if the orbit is periodic or in equilibrium. Discarding the still state, so that $x(t) \neq k$, we define a function $t(x)$ such that

$$t : X \mapsto \mathfrak{R} \quad (5)$$

$$t(x) := \min\{t \mid x(t) = x\} \quad (6)$$

We have that may be

$$t(x(t_0)) \neq t_0 \quad (7)$$

but this doesn't change the properties of F as we see from (3) and we find

$$x(t_0 + \Delta t) - x(t_0) = F(t(x(t_0)) + \Delta t) - F(t(x(t_0))) \quad (8)$$

We can choose F so that $F(t(x(t_0))) = 0$, and it follows that

$$\begin{aligned} x(t_0 + \Delta t) - x(t_0) &= F(t(x(t_0)) + \Delta t) \\ x(t_0 + \Delta t) &= x(t_0) + F(t(x(t_0)) + \Delta t) \end{aligned} \quad (9)$$

If we choose a fixed Δt we can define

$$M : X \mapsto X \quad (10)$$

$$M(x) := x + F(t(x) + \Delta t) \quad (11)$$

and write

$$x(t + \Delta t) = M(x(t)) \quad (12)$$

Moreover if we call

$$\begin{aligned} x_0 &:= x(t_0) \\ x_1 &:= x(t_0 + \Delta t) \\ x_2 &:= x(t_0 + 2\Delta t) \\ &\vdots \\ x_n &:= x(t_0 + n\Delta t) \end{aligned} \quad (13)$$

²This has sense since $x(t)$ is defined by f via (1).

we can study the trajectory in a countable number of points x_n . The evolving rule is

$$x_{n+1} = M(x_n) \tag{14}$$

where M maps a state of the system into its successor after time Δt . In fact M is called a *map*. I have to note that our construction of M is self-recursive, so may not be applicable in practice. There are other ways to build a map M from a trajectory $x(t)$, like the *Poincaré section* method that we'll see later.

2 Definition

Let's try to get some results from what we have done above. If state of a system can be fitted into a vector $x \in X$, then we have seen that the evolution of a system can be predicted using different ways:

- a differential equation;
- its continuous solution;
- a map.

We can define a *dynamical system* as:

- a space of possible configurations;
- a rule to evolve such configurations.

As an example we can look at one of the physicians' classic: the harmonic oscillator. Being r the position, the differential equation is

$$\ddot{r} = -\omega^2 r \tag{15}$$

The analytical solution for this equation is known to be

$$r = A \cos(\omega t + \phi) \tag{16}$$

where A , the amplitude, and ϕ , the phase, can be chosen freely. The state of the system, indicated by position r and speed v , can be fitted into the vector

$$x = \begin{pmatrix} r \\ v \end{pmatrix} \tag{17}$$

The differential equation, put into a first-order form (ODE), is now

$$\dot{r} = v \tag{18}$$

$$\dot{v} = -\omega^2 r \tag{19}$$

and in vectorial notation

$$\frac{d}{dt} \begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} r \\ v \end{pmatrix} \quad (20)$$

We can integrate this in respect to the time in an interval $(t_0, t_0 + \Delta t)$ and obtain

$$\Delta r(\Delta t) = \int_{t_0}^{t_0+\Delta t} v(\tau) d\tau \quad (21)$$

$$\Delta v(\Delta t) = -\omega^2 \int_{t_0}^{t_0+\Delta t} r(\tau) d\tau \quad (22)$$

We can remove explicit dependences from time t_0 , but not from the initial conditions $r(t_0)$ and $v(t_0)$

$$\Delta r(\Delta t) = \int_{t_0}^{t_0+\Delta t} [v(t_0) + \Delta v(\tau - t_0)] d\tau \quad (23)$$

$$= v(t_0)\Delta t + \int_0^{\Delta t} \Delta v(\tau) d\tau \quad (24)$$

and the same with Δv

$$\Delta v(\Delta t) = -\omega^2 \int_{t_0}^{t_0+\Delta t} [r(t_0) + \Delta r(\tau - t_0)] d\tau \quad (25)$$

$$= -\omega^2 \{r(t_0)\Delta t + \int_0^{\Delta t} \Delta r(\tau) d\tau\} \quad (26)$$

so that

$$r(t_0 + \Delta t) = r(t_0) + \Delta r(\Delta t) \quad (27)$$

$$v(t_0 + \Delta t) = v(t_0) + \Delta v(\Delta t) \quad (28)$$

but, as you can expect, this definition is self-recursive, being the result of our method, and so not applicable in practice.

We can avoid this problem using numerical approximations to resolve the integrals, since they are computed on an interval of amplitude Δt which we can choose to be enough small to considerate r and v constants during the interval. This technique used to solve an ODE is known as *explicit euler*. Or we can use other finer techniques, but that's another story...

3 Studing

When studing a dynamical system the first thing that we can think is to take an initial condition, $x(t_0)$ or x_0 , and concentrate on a single trajectory, being

it continuous or discrete. But another interesting possibility is to concentrate on the evolution of a *set* of initial conditions, i.e. a subset $Y \subset X$, and look at its behaviour under the evolving rule. Subsets of X evolve and go into other subsets of X . So we can write, using a map M and a the set of initial conditions Y_0 ,

$$\begin{aligned} Y_1 &= M(Y_0) \\ &\vdots \\ Y_{n+1} &= M(Y_n) \end{aligned} \tag{29}$$

This is conceptually a great change of view.

Let's go on with the example of the harmonic oscillator. As we have seen the space X of the possible configurations is formed, in this case, by couples of position and speed (r, v) , so $X = \mathfrak{R} \times \mathfrak{R}$.

In a 2-d plan where r is on the X axis and v is on the Y axis each point can be thought as an initial condition for the system. This plan is called *phase space* and can be built for every dynamical system. Clearly it has always the same dimension of the vectorial space X .

If we choose to begin with time $t_0 = 0$, then the initial condition $(r(0), v(0))$ gives the values of A and ϕ

$$r(0) = A \cos(\phi) \tag{30}$$

$$v(0) = A\omega \sin(\phi) \tag{31}$$

so that

$$A = \sqrt{(r(0))^2 + (v(0)/\omega)^2} \tag{32}$$

$$\phi = \arctan \frac{v(0)/\omega}{r(0)} \tag{33}$$

and the evolution is given by

$$r(t) = A \cos(\omega t + \phi) \tag{34}$$

$$v(t) = A\omega \sin(\omega t + \phi) \tag{35}$$

It's clear that a set of initial conditions $Y \subset X$ on the phase space rotates and eventually stretches along the Y axis with time, the latter depending on the value of ω . In we choose $\omega = 1$ this is a perfect rotation.

We can ask if there are sets which are invariant under the evolution. A set with this property is called a *manifold*

$$Y \subseteq M(Y) \tag{36}$$

For the harmonic oscillator these manifolds are ellipses, or circles if $\omega = 1$. But thinking twice we see that these sets are also the periodic orbits of this system. In fact (36) tells us that if a point is in a manifold, it will remain there forever, and that is true for all periodic orbits.

Here I used a map, but we can always pass from a continuous trajectory to a map. Let's call the trajectory

$$T := \{x(t)\} \tag{37}$$

we have

$$T \subset X \tag{38}$$

and we can imagine this subset in the phase space.

Now let's lose some time on what we call a *surface*. In our 3-d space it is a 2-d subset. Now if X has a dimension $\dim(X) \geq 1$, S is a surface of X if

$$S \subset X \tag{39}$$

and

$$\dim(S) = \dim(X) - 1 \tag{40}$$

If we choose S so that the intersection between S and T is not empty and numerable, we may call

$$\begin{aligned} t_1 &:= \min\{t \mid x(t) \in S\} \\ t_2 &:= \min\{t > t_1 \mid x(t) \in S\} \\ t_3 &:= \min\{t > t_2 \mid x(t) \in S\} \\ &\vdots \\ t_n &:= \min\{t > t_{n-1} \mid x(t) \in S\} \end{aligned} \tag{41}$$

Defining

$$\begin{aligned} x_1 &:= x(t_1) \\ x_2 &:= x(t_2) \\ x_3 &:= x(t_3) \\ &\vdots \\ x_n &:= x(t_n) \end{aligned} \tag{42}$$

$$\tag{43}$$

we have a succession extracted from the trajectory, and we can think of it as the iteration of map. This method is called the *Poincaré' section*.

As an example of it let's see what it can do with the harmonic oscillator. If we choose S so that

$$S := \{(r, v) \mid r = 0\} \tag{44}$$

then

$$\begin{aligned} x_1 &= (0, v_{max}) \\ x_2 &= (0, -v_{max}) \\ x_3 &= (0, v_{max}) \\ x_4 &= (0, -v_{max}) \\ &\vdots \\ x_n &= (0, (-1)^{n+1}v_{max}) \end{aligned} \tag{45}$$

so we have a succession of period 2. This means that

$$x_n = x_{n+2} = M(M(x_n)) \tag{46}$$

that we can write as

$$x_n = x_{n+2} = M^2(x_n) \tag{47}$$

In the general case, if a map M has period p , then each of the p elements of the period is a fixed point for the map M^p .

If T is periodic, then each map M derived from T is periodic and can be set to have period 1 with this method.

4 Chaos

What is chaos? What do we mean about it?

To be continued...